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A simple and general axiomatization of average utility maximization for infinite streams

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ABSTRACT

This paper provides the most general preference axiomatization of average utility (AU) maximization over infinite sequences presently available, reaching almost complete generality. The only restriction is that all periodic sequences should be contained in the domain. Infinite sequences may designate intertemporal outcomes streams where AU models patience, welfare allocations where AU models fairness, or decisions under ambiguity where AU models complete ignorance. As a methodological contribution, this paper shows that infinite-dimensional representations can be simpler, rather than more complex, than finite-dimensional ones. Infinite dimensions provide a richness that may be convenient rather than cumbersome. In particular, (empirically problematic) continuity assumptions are not needed in our axiomatization. Continuity is optional.

1. Introduction

Many authors have argued for "fair" average utility (AU) maximization, as a normative objection to impatience and discounted utility (da Volterra 1574; Elster 1986 pp. 10–11; Jevons 1871; Pigou 1920; Sidgwick 1874; Weinstein 1993). AU maximization has also gained popularity for fair social welfare evaluations and decisions under complete ignorance (Gravel et al. 2018). It is central in Laplace's (1796) principle of insufficient reason and is used in the folk theorem for repeated games (Peters 2015 §7.2.2). As an important and widely applicable decision model, AU has received numerous axiomatizations for the finite-dimensional case.[1]

This paper considers AU for infinitely many timepoints (dimensions). Here, mathematical and philosophical problems can arise as limits may diverge and choice paradoxes may arise. For example, patience or fairness may be irreconcilable with strong Pareto for welfare evaluations (Diamond 1965). Similarly, stochastic dominance may be violated under uncertainty (Wakker 1993). The two-envelope paradox and Dubin's paradox are among the paradoxes for uncertainty.[2] These problems have intrigued researcher for a

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[1] These axiomatizations were central in Blackorby et al. (1977), Gravel et al. (2012), and Kothiyal et al. (2014). Every axiomatization of additive $\sum_{j=1}^{n} V_j(x_j)$ (Debreu 1960; Gorman 1968; Krantz et al. 1971), including expected utility $\sum_{j=1}^{n} p_j U_j(x_j)$ (Gul 1992; Chew and Karni 1994) with fairness (symmetry) added gives AU. Further, quasilinear means from mathematics are constant-equivalence functions of AUs, giving many more axiomatizations (Aczél 1966 pp. 151 & 240; Münnich et al. 2000).

[2] For the former, see Kraitchik (1953), Nalebuff (1989), and Yi (2013). For the latter, see Howson (2014). Both concern the (im)possibility to have a uniform probability distribution over the natural numbers, which underlies AU.

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long time, and many solutions have been discussed, often by restricting the domain of preference or relaxing completeness of preference.

This paper provides a preference axiomatization of AU maximization over infinite sequences of great and almost complete generality. For instance, unlike all preceding axiomatizations, we do not need the assumption of continuity, which is less innocuous than it may seem as explained below. Similar to previous studies, we adopt the approach of domain restriction. Our only restrictive assumption is that all periodic sequences (defined later) should be contained in the preference domain. This is considerably less stringent than what has been assumed in all preceding axiomatizations of AU.[3] Other than this restriction, we obtain complete generality regarding preference relations, outcome sets, utility functions, and preference domains. Thus, we provide maximal possibilities for reconciling desirable properties and detecting mathematical problems.

Although we do not need continuity, we can easily incorporate it when desired. Thus, Observation 6 adds continuity and provides the most general axiomatization of continuous AU in the literature. To obtain our results, we solve two problems for preference axiomatizations of AU, explained next.

The first problem occurs already in finite-dimensional preference domains. If these domains are even finite (finitely many elements), they are usually too coarse to allow for precise quantitative representations. This complicates the mathematical analysis, to the extent that preference axiomatizations for finite domains are usually unknown. Therefore, many researchers commonly resort to the simplifying assumption that the preference domain is a continuum, as for instance throughout Adler (2019), Blackorby et al. (2005), and numerous papers. That is, they add extra structural richness to simplify the analysis. Quantitative representations can then be precisely identified using continuity assumptions,[4] and the mathematical analysis becomes tractable. Thus, Chambers et al. (2021) wrote: "Continuity is a necessary regularity condition; without it, no meaningful inferences can be made with any finite amount of data."

Continuing our explanation of the first problem, the assumption of continuity (w.r.t. a continuum domain) comes with empirical cost. To explain the problematic empirical status of continuity, we first note that, in isolation, continuity can never be verified or falsified by a finite number of observations. In this sense, it has no empirical content. However, this is not a problem. It rather is an advantage. We know exactly what we are doing empirically when assuming continuity in isolation: nothing! Thus, several authors have argued that continuity is empirically harmless. For instance, Arrow (1971 p. 48) wrote: "The assumption of Monotone Continuity seems, I believe correctly, to be the harmless simplification almost inevitable in the formalization of any real-life problem", a point supported by Drèze (1987 p. 12) and Thomson (2001 §4.1.3 p. 338). Unfortunately, those suggestions conceal serious underlying problems. If other axioms are assumed, then continuity is not empirically vacuous. It can then add empirical content to those other axioms. This would not be a problem if we knew what that empirical content is, but for continuity in our context we do not know that exactly. That is, we do not exactly know how to verify or falsify our axiom set including continuity from finitely many observations. We thus lose track of what we are doing when continuity and other axioms are assumed together. Many authors have pointed out this problem.[5]

The second problem occurs with infinite-dimensional domains (Asheim 2010 §4.2; Pivato 2014 p. 35). With one exception cited later, authors invariably used the axioms for finite-dimensional axiomatizations including restrictive continuity to handle finite-dimensional subdomains. They then added axioms to extend the representation to the infinite dimensions. This way, the extra richness of infinite dimensions is an extra problem. Further, continuity conditions are more complex for infinite dimensions.

We provide a solution for the two aforementioned problems. In this solution, the extra richness of infinite dimensions is not an extra problem. Instead, it simplifies the analysis and even removes the first problem, leaving no problem at all. Key is that the richness of infinite dimensions can substitute for the richness of continuity.

Theorem 3 presents our solution, assuming infinitely many dimensions and not requiring any continuity. The theorem is obtained by extending an AU axiomatization of Kothiyal et al. (2014) to infinite dimensions (see the Appendix). Kothiyal et al. considered finite streams but allowed any finite length. Their domain can be called half-infinite-dimensional. Observation 4 illustrates our approach by showing how the infinite dimensions enable us to uniquely capture utility without needing continuity. To our best knowledge, Pivato (2014 p. 56) is the only paper that shares our observation that infinitely many dimensions may be a convenience rather than a problem.

Observation 5 shows that our new methodology can constructively obtain a preference domain where AU is maximized. This result allows us to easily identify any domain where AU maximization does not encounter any of the aforementioned mathematical problems or paradoxes. In particular, a reconciliation is obtained for discrete outcome sets, a case not considered before in the literature.

2. Basic definitions

Γ denotes a set of *outcomes*, with generic notation $\alpha, \beta, \ldots$ (or f_j, g_j; see later). Outcomes can be quantitative or not, and Γ can be finite or infinite. *Streams* are infinite sequences $f = (f_1, f_2, \ldots)$ of outcomes, with generic notation $f, g, \ldots$. They can be welfare allocations over individuals where AU captures fairness, time profiles where AU captures patience, gambles on states of nature where AU captures extreme ambiguity (complete ignorance), commodity bundles, and so on. Our results can be applied to all these contexts. We interpret f_j as the *outcome* for *generation j*, combining welfare and intertemporal considerations. F, the *preference domain*, denotes a subset of the set of all streams, further specified later.

[3] See Fishburn and Edwards (1997), Harvey (1986), Lauwers (1998), Marinacci (1998), Pivato (2022), and Rébillé (2007).

[4] It is understood here that the right scale type, such as an equivalence class of interval scales (defined later), is then uniquely identified.

[5] See Khan and Uyanik (2021), Kothiyal et al. (2014), Krantz et al. (1971 §9.1), Pfanzagl (1968 §6.6), Pivato (2014 p. 32), and Wakker (1988).

By $\succcurlyeq$ we denote a binary relation on F, the *preference relation*. We call $\succcurlyeq$ a *weak order* if it is *complete* ($f \succcurlyeq g$ or $g \succcurlyeq f$ for all $f, g \in F$) and transitive. The notation $\succ, \preccurlyeq, \prec, \sim$ is as usual. *Average utility* (*AU*) holds on a subset $F' \subset F$ if there exists a *utility function* $U : \Gamma \to \mathbb{R}$ such that the *average utility* $\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} U(f_j)$ $(= AU(f))$ exists[6] for every stream f in F' and *represents preference* on F', i.e., $f \succcurlyeq g \Leftrightarrow AU(f) \geq AU(g)$ on F'. Note that the ordering of natural numbers and indexes is essential in the definition of AU. Therefore, AU and our analysis can only be applied when such an ordering is naturally available. For example, this occurs for discrete timepoints and for consecutive generations, but not if the set of indexes is the set of rational numbers.

We identify outcomes α with constant streams $(\alpha, \alpha, \ldots)$. Hence, $\succcurlyeq$ also applies to outcomes. To avoid triviality, we assume throughout that Γ contains at least two nonindifferent outcomes. *Monotonicity* holds if $f \succcurlyeq g$ whenever $f_j \succcurlyeq g_j$ for all j. *Strong Pareto* holds if $f \succ g$ whenever $f_j \succcurlyeq g_j$ for all j and $f_j \succ g_j$ for some j. *Weak Pareto* holds if $f \succ g$ whenever $f_j \succ g_j$ for all j.

Fairness means that, for a given set of permutations of the indexes, none affects preference. For precise definitions, the set of permutations should be specified. Other names for fairness used in the literature include anonymity, patience, impartiality, (intergenerational) equity, intergenerational neutrality, or symmetry. Fairness is typically violated under discounting with impatience, with early generations privileged over later ones. By treating infinitely many generations the same, mainly fairness implies that any finite number of generations is negligible. Only the long run matters.

Fairness is often incompatible with strong Pareto (Diamond 1965; Lauwers 1998; Neyman, 2023 Fact 12; Weymark 1995). This has been the topic of many studies, where different strategies of reconciliation have been examined. Asheim (2010) and Petri (2019) provided surveys. Because our model can do without continuity, a new reconciliation possibility immediately emerges for weak Pareto: if we have a discrete utility range and there exists $\varepsilon > 0$ such that any two non-indifferent outcomes have a utility distance of at least ε. Then under AU, indeed, $f \succ g$ whenever $f_j \succ g_j$ for all j. Another solution, widely studied, is to limit the set of preferences considered, e.g., by relaxing completeness (see §6) or, our approach, by considering particular subdomains (p-streams, defined later).

3. Average utility for periodic streams

For a finite sequence $(x_1, \ldots, x_n)$ of outcomes, called *generator*, the *periodic extension*, denoted $[x_1, \ldots, x_n]$, is the stream $x = (x_1, \ldots, x_n, x_1, \ldots, x_n, x_1, \ldots)$. That is, $x_{jn+i} = x_i$ for all $i \leq n$ and j. Periodic extensions are also called *periodic streams* (*p-streams*). We call n the *length* (of the periodic extension/stream or the generator), denoted $||x||$, bearing in mind that periodic streams are infinite sequences.[7] For each outcome α, $[\alpha] = (\alpha, \alpha, \ldots)$ has length $||[\alpha]|| = 1$. F^p denotes the set of *all* periodic sequences, with generic notation $x, y, \ldots$.

The "infinitistic" AU of a p-stream is the "finitistic" average utility of its generator. This suggests that p-streams can combine the intuitive convenience of finite-dimensional simplicity and the mathematical convenience of infinite-dimensional richness. This insight underlies the analysis of our paper. We impose conditions as much as possible only on F^p, where they are simple. Accordingly, we first derive our results on F^p. Observation 1 will achieve complete generality of AU maximization there, extending Kothiyal et al. (2014 Theorem 7) to infinite dimensions.

For every preference condition C of $\succcurlyeq$, *periodic C* (*p-C*) refers to that condition when restricted to F^p, as in p-weak ordering, p-monotonicity, and so on. We define *periodic fairness* (*p-fairness*) to imply that any permutation of the x_is in the generator of any p-stream $[x_1, \ldots, x_n]$ leaves the stream indifferent. This permutation, finitistic in spirit, does involve infinitely many generations. For example, interchanging $i = 1,2$ of the generator means interchanging every $j \times n + 1$ and $j \times n + 2$ of the p-stream.

P(eriodic)-independence holds if

$$[c_1, x_2, \ldots, x_n] \succcurlyeq [c_1, y_2, \ldots, y_n] \Rightarrow [d_1, x_2, \ldots, x_n] \succcurlyeq [d_1, y_2, \ldots, y_n] \tag{1}$$

Note that this condition involves identical outcomes for generations $n+1$, $2n+1$, and so on. By fairness, the condition implies that preferences between p-streams of the same length are also independent of common 2nd, 3rd, …, and n^{th} dimensions of the generator and, by repeated application, of any number of common dimensions. That is, the condition amounts to regular separability for the generators. Asheim and Zuber (2014), Rébillé (2007), and Zuber and Asheim (2012) reconciled fairness and Pareto by using rank-dependent weakenings of separability.

AUs, as do all real numbers, have to satisfy an Archimedean axiom. To prepare for a corresponding preference condition, a notation: for p-streams x, y, $[nx, my]$ denotes the periodic stream extending n times the (finite) generator of x followed by m times the (finite) generator of y.[8] For example, if $x = [x_1, x_2]$ and $y = [y_1, y_2]$, then $[2x, 3y] = [x_1, x_2, x_1, x_2, y_1, y_2, y_1, y_2, y_1, y_2]$. We suppress $n = 1$ and $m = 1$, as in $[x, 4y] = [1x, 4y]$. This paper uses the "for all" symbol $\forall$ and the "there exists" symbol $\exists$. The *p-Archimedean axiom* holds if:

$$\forall x, y, v, w \in F^p \text{ with } ||x|| = ||y||,\ ||v|| = ||w||,\ x \succ y,\ \exists n \in \mathbb{N} :\ [nx, v] \succcurlyeq [ny, w] \tag{2}$$

[6] In our terminology, existence implies being real-valued. The average is sometimes called the Cesàro limit.

[7] For a given p-stream, any finite replication of its generator can be taken as another generator, and in this sense, the length depends on the generator specified. This will never cause confusion in our analysis. Formally, for unique definitions, we could commit to minimal lengths and generators, with, for instance, (x_1, x_2) the generator of $[x_1, x_2, x_1, x_2]$, e.g., to obtain a unique notation $[nx, my]$ defined below. P-streams are isomorphic with the set of finite sequences of varying lengths modulo the formal identity just mentioned, corresponding with a condition for finite sequences called replication-invariance (Kothiyal et al. 2014; see the proof of our Observation 1).

[8] Formally, this notation requires specification of the generators chosen. This requirement will never cause problems in our analysis, where the choice of generator will never matter. For simplicity of presentation, we leave the chosen generator implicit.

That is, no matter how disadvantageous $v \prec w$ is, it can be overcome by sufficiently many (n) advantages $x \succ y$.

Observation 1. Assume that the preference domain F contains all periodic streams F^p. Then

(i) AU holds on F^p

$\Leftrightarrow$

(ii) $\succcurlyeq$ satisfies:
1. p-weak ordering;
2. p-Archimedeanity;
3. p-fairness;
4. p-independence.

In the above observation, p-monotonicity is implied by the other conditions. Strong Pareto is also implied on F^p by the other conditions. The simplicity of the necessary and sufficient conditions in Observation 1 and in Kothiyal et al. (2014 Theorem 7) is in stark contrast to the very complex necessary and sufficient conditions needed for general finite-dimensional structures (Jaffray 1974).[9]

We present the proof of the following observation in the main text because it illustrates how the richness of infinitely many timepoints brings richness of utility without requiring any continuity. This observation is key to our approach, exploited in the next section.

Observation 2. $AU(F^p)$ is dense within its convex hull.

Proof. Consider p-streams x, y with lengths $||x||$ and $||y||$. We have $AU([||y||x, ||x||y]) = \frac{1}{2}AU(x) + \frac{1}{2}AU(y)$. Apply this repeatedly.

4. Average utility for general streams

We now turn to general, nonperiodic streams. Pivato (2014) considered representations for such streams using nonstandard real numbers. He generalized Kothiyal et al. (2014), using intuitive axioms like those in our Observation 1. Several other papers considered reconciliations of fairness and Pareto using general orderings with multi-utility representations and incompleteness that cannot be represented by real numbers (Basu and Mitra 2007; Bossert et al. 2007; Khan and Stinchcombe 2018). We stick to representations by standard real numbers and see how far we can go. Our decision-theoretic task is to obtain axiomatizations that use only conditions directly in terms of $\succcurlyeq$.

P-streams will be used to calibrate the other streams. F^p is rich enough to serve this purpose. We therefore assume, implicitly throughout the text and explicitly in theorems, that F contains all p-streams. We next define the required preference conditions. To explain their intuition, we will claim several implications of AU on F^p. These are all proved in the proof of Theorem 3.

We first rule out infinite AU values. To prepare, the outcome set Γ is *unbounded above* if

$$\forall \beta \prec \gamma \in \Gamma,\ \forall n \in \mathbb{N},\ \exists \delta \succ \gamma \in \Gamma: \quad [\delta, n\beta] \succ [\gamma, n\gamma] \tag{3}$$

We used the notation $[\gamma, n\gamma]$ instead of γ (which is the same) for clarification. The condition implies that, no matter how many times (n) we receive the drawback of β instead of γ, there is an outcome δ so good and exceeding γ so much that receiving it once instead of γ is enough to overcome those n drawbacks. Under AU on F^p it is necessary and sufficient for U to be unbounded above. Similarly, Γ is *unbounded below* if

$$\forall \gamma \succ \beta \in \Gamma,\ \forall n \in \mathbb{N},\quad \exists \alpha \prec \beta \in \Gamma: \quad [\alpha, n\gamma] \prec [\beta, n\beta] \tag{4}$$

Under AU on F^p, the condition is necessary and sufficient for U to be unbounded below. Stream f is *value-unbounded above* (*v-unbounded above*), if Γ is unbounded above and

$$\forall \gamma \in \Gamma\ \exists n \in \mathbb{N}: \quad [f_1, \ldots, f_n] \succcurlyeq \gamma \tag{5}$$

Under AU on F^p, the condition holds if and only if $limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}U(f_j) = \infty$. Stream f is *v-unbounded below* if Γ is unbounded below and

$$\forall \alpha \in \Gamma\ \exists n \in \mathbb{N}: \quad [f_1, \ldots, f_n] \preccurlyeq \alpha \tag{6}$$

Under AU on F^p, the condition holds if and only if $liminf_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}U(f_j) = -\infty$. Stream f is *v-bounded* if it is neither v-unbounded above nor below. Under AU on F^p, f is v-bounded if and only if $liminf_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}U(f_j)$ and $limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}U(f_j)$ are finite. Thus, v-boundedness of f need not imply boundedness of f itself.

We next rule out v-bounded streams f whose AU is not well-defined. Stream f is *stable* if for all p-streams $x \succ y$ there exists $N \in \mathbb{N}$ such that $[f_1, \ldots, f_n] \preccurlyeq x$ for all $n \geq N$ or $[f_1, \ldots, f_n] \succcurlyeq y$ for all $n \geq N$. That is, we must be able to decide whether f is below x or above y, or

[9] Kothiyal et al. (2014) considered finite sequences, but their domain was "half-infinite-dimensional" in the sense that there was no upperbound to the lengths of sequences.

possibly both. Under AU on F^p, stability holds if and only if $liminf_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}U(f_j) = limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}U(f_j)$. That is, if the liminf is strictly smaller than the limsup, so that $\frac{1}{n}\sum_{j=1}^{n}U(f_j)$ keeps on fluctuating through the interval between them, then, because of the denseness of $AU(F^p)$, we can find p-streams x, y within this interval that reveal this fluctuating character. For this revelation, we only use observable preferences between auxiliary p-streams.

If we have AU on F^p then AU is well-defined and finite for v-bounded stable f. Finally, we ensure that AU represents the preference relation. *P-denseness* holds if, for all streams f, g,

$$(f \succ g) \Leftrightarrow (\exists x, y \in F^p, N \in \mathbb{N} : \ \forall n \geq N : \ [f_1, \ldots, f_n] \succ x \succ y \succ [g_1, \ldots, g_n]) \tag{7}$$

Under AU on F^p, the condition ensures that for every strict preference $f \succ g$ we can get a strict p-stream preference in between f and g because of the denseness of $AU(F^p)$. More precisely, we can get them in between the beginning n outcomes of f and g for all n far enough into the future. For this revelation, we again only use observable preferences between auxiliary p-streams. In the condition, getting $x \succ y$ in between precludes infinitesimal strict preferences $f \succ g$. In the following theorem, an *interval scale* is unique up to level and unit (positive affine transformations).

Theorem 3. *Assume that the preference domain F contains all periodic streams F^p. Then*

(i) AU holds on F.

$\Leftrightarrow$

(ii) $\succcurlyeq$ satisfies:
1. weak ordering;
2. p-Archimedeanity;
3. p-fairness;
4. p-independence;
5. all $f \in F$ are v-bounded;
6. all $f \in F$ are stable;
7. p-denseness.

Further, if (i) holds, then U is an interval scale.

In the theorem, monotonicity is again implied by the other conditions. Strong Pareto need not always hold, and this depends on the domain F. Because our domain F is the most general for AU available in the literature as yet, it provides maximal possibilities of reconciliations through domain restriction.

5. Further results

The following observation shows in a direct manner that the infinitely many timepoints provide enough richness to uniquely calibrate U. This result further illustrates why we do not need continuity. The result is reminiscent of probability equivalent utility measurements in expected utility (Baucells and Villasís 2015; Mosteller and Nogee 1951). A proof of Observation 1 alternative to the one in this paper could have been obtained by defining a mixture operation with rational mixing weights on p-streams and then using mixture space techniques. Observation 4 immediately follows from substitution.

Observation 4. Assume AU and $F^p \subset F$. Assume $\gamma \succ \beta \succ \alpha \in \Gamma$. Then the preference between $[m\gamma, n\alpha]$ and β corresponds exactly with the ordering of $\frac{m}{m+n}U(\gamma) + \frac{n}{m+n}U(\alpha)$ and $U(\beta)$. If we scale $U(\gamma) = 1$ and $U(\alpha) = 0$, then this result uniquely determines $U(\beta)$. In general, it uniquely identifies the interval scale U.

Our preference conditions allow for a constructive definition of preference domains where AU can hold. To explain this point, we assume weak ordering. We start with the set F^p. We then seek to extend the AU representation step by step, each time verifying if streams to be added satisfy all required conditions. That is, they must be v-bounded and stable, and the right-hand side of Eq. (7) should hold for every newly added preference $f \succ g$. We developed our preference conditions so that only p-streams are invoked as auxiliary tools. This shows that p-streams provide a convenient calibration tool for constructing domains where mathematical problems and inconsistencies can be avoided. This somewhat informal result is displayed next.

Observation 5. The set of periodic streams offers sufficient calibration possibilities to constructively define any preference domain for AU.

As explained, continuity of utility is optional in our approach. The following observation covers connected topological spaces Γ, which includes all intervals, all convex subsets of commodity spaces, and many mixture-closed sets of probability distributions over prizes. Fishburn and Edwards (1997) also assumed connected topological outcome spaces. All other references made more restrictive topological assumptions.

Observation 6. Assume AU on F which contains all periodic streams F^p with the outcome space Γ a connected topological space. Then U is continuous if and only if $\succcurlyeq$, restricted to p-streams $[f_1, f_2]$, is continuous when taken as a binary relation on $\Gamma \times \Gamma$ endowed with the product topology.Of course, continuity of preference above can be strengthened to hold for any p-streams of any length and by specifying proper infinite-dimensional continuities.

Appendix B shows that all axioms in Theorem 3 are independent. For each axiom, an example is given where that axiom is violated but all other axioms are satisfied. Thus, none of the axioms is redundant in the sense of being implied by the others.

6. Related literature

Fishburn and Edwards (1997) axiomatized AU but only for pairs of streams that differ on no more than finitely many timepoints. Then the long run does not matter. The model is essentially finite-dimensional and too restrictive for most purposes. It contains no preferences between periodic streams.

Pivato (2022) is closest to our paper. He was the first to axiomatize AU for (truly) infinite sequences. His outcome set is a connected metric space and U is continuous. His preference domain contains (roughly) the closure of all "regular totally bounded" sequences. Regular means that limiting frequencies exist. This requirement incorporates all our periodic sequences. All permutations on periodic streams that our fairness condition involves belong to his Lévy group, and, hence, his Γ-invariance implies our fairness. His AU does not satisfy strong Pareto. He did not have to impose an Archimedean axiom because it is implied by continuity. Similarly, many other papers in the literature used continuity to imply the Archimedean axiom. Relatedly, Chew and Karni (1994) showed that Gul's (1992) continuity can be dispensed with by using an Archimedean and a solvability axiom, both implied by Gul's continuity.[10]

We next discuss some papers that did not exactly axiomatize AU for infinite sequences, but instead considered close generalizations and/or modifications for infinite sequences. Harvey (1986, Theorem 2a and 2′a) considered the maximization of sums of utilities rather than averages. To have those sums finite, he specified a status quo outcome α^* with utility 0, and only considered streams that, roughly, converge to α^* so strongly that the sum of utilities is defined (his Definition 9). They all have $AU = 0$, and his model can be considered to maximize infinitesimal AU. But it shares most characteristics with AU. In particular, it satisfies the preference conditions in our Theorem 3 except p-denseness. On his domain, strong Pareto ("strict increasingness") and fairness ("time neutrality") are reconciled. The main difference with our model is that his domain, besides not including periodic streams, is very restricted, with the long run never mattering, and always ending up at α^*.

Lauwers' (1998) theorems took a representing functional instead of a preference relation as primitive. His domain was the set of all bounded real-valued sequences. He assumed linear utility (($U(\alpha) = \alpha$), implied by linearity of the functional, and supnorm continuity. He also incorporated nonstable streams with representations between the limsup and the liminf of AU (his Theorem 2). His anonymity immediately implies p-fairness. It cannot be reconciled with strong Pareto.

In Marinacci (1998), the outcome set consists of all simple probability distributions over a set of prizes. Expected utility is maximized over outcomes. That is, the outcome set is convex and utility is linear. The preference domain consists of all bounded streams. Marinacci assumed continuity with respect to probabilistic mixing of the outcomes and axiomatized liminf AU representations or, more generally, their Polya extensions, using axioms similar to those from multiple priors models in decision under ambiguity. He next added a time invariance axiom that implies AU maximization. His patience implies our p-fairness. He did not consider reconciliation with strong Pareto.

Rébillé (2007) assumed real-valued outcomes and linear utility and considered the domain of all bounded sequences. He considered several generalizations of AU such as discounted rank-dependent forms, but did not derive AU itself.

Neymann (2023) also assumed real-valued outcomes and linear utility and considered only bounded sequences. His Theorem 2 characterized AU mainly by linearity and what he called, in deviation from common terminology, patience (roughly, linear patience restricted to constant streams).

Strong Pareto and fairness can be trivially reconciled by excluding all problematic preference situations, leading to a relaxation of completeness. Several papers studied such relaxations, but with the purpose of obtaining nontrivial results with a rich and interesting domain of preference situations. They often incorporated preferences where f and g have undefined AU values, with representations between *limsup AU* and *liminf AU*. These papers all assumed linear utility or, equivalently, that outcomes are utils, and they focused on bounded sequences or overtaking criteria (Gale 1967; Jonsson and Voorneveld 2018; Svensson 1980). Other papers considered fairness conditions so much weaker that they do not conflict with strong Pareto, including Lauwer's (2012) fixed-step anonymity. Mitra and Basu (2007) showed that such fairness restrictions must satisfy cyclicity and conditions regarding mathematical group-operations. Our p-fairness satisfies these conditions and is weaker than the aforementioned fairness conditions. In return, the cited papers could handle more general representations than AU.

All aforementioned studies assumed continuous and mostly even linear utility. Whenever the long run was relevant, all p-streams were included in the domain. Focusing on AU representations as defined in §2, our result is uniformly the most general in the sense that our assumptions are satisfied for every such representation in the literature but not the other way around. Besides (1) continuity (always more restrictive than in our Observation 6) all published results (2) focused on bounded streams whereas we can handle all unbounded streams as long as their AU is finite; (3) assumed rich domains of all streams that are bounded and satisfy some regularity assumptions, whereas we allow for almost any kind of subdomain; (4) used more permutations in their fairness than in our p-fairness. Our result does not generalize the existing results in a logical sense. That is, the assumptions in existing results do not imply our assumptions in an elementary manner—to our best knowledge—and in this sense are not corollaries of our results. Further, most other studies extended the AU representation to streams for which AU is not defined or infinite, and studied other properties than considered in this paper. Thus, the existing results remain of independent interest.

[10] We thank an anonymous referee and associate editor for bringing in these two references. Solvability, like continuity, brings in unknown empirical implications, but less so than continuity because it is less restrictive.

7. Discussion

Many papers in welfare theory take individual utilities ("utils" or "welfare") as given or, equivalently, assume real-valued outcomes and linear utility for all individuals. However, utility is often nonlinear with individual utilities subjective and not directly observable. We took the more general approach of allowing for general utility, to be revealed from preferences. AU does assume the same utility function for different js. That is, the same outcome gives the same utility for different generations. This is a common assumption for individual intertemporal (non)discounted utility, but for welfare evaluations there is interest in generation-dependent utility. Harvey (1986 Theorem 9) and Wakker and Zank (1999) provided models with generation-dependent utility, but they heavily used continuity. We leave such generalizations of our results to future work.

The Archimedean axiom is a technical axiom, like continuity. Accordingly, one could be concerned about a similarly problematic empirical status. However, it has been shown that Archimedean axioms do not have such problems in several situations: finitely many observations verify or falsify a set of other axioms if and only if they do so with the Archimedean axiom added (Luce et al., 1990 Theorem 21.21). The axiom then has no empirical content and is innocuous. We do not know to what extent such a result holds for the theorems in this paper. However, these problems of empirical status are smaller than for the more restrictive continuity axioms and cannot be avoided anyhow because the Archimedean axiom is necessary for any AU representation.

The three axioms used to extend AU to non-periodic streams involve "there exist" quantifiers and their negations,[11] and share the drawbacks of all axioms of this kind. Whereas it is common in the literature to use continuity axioms to ensure that integrals are well-defined and finite (and those axioms bring extra restrictions), our three axioms are not only sufficient but also necessary for AU. That is, they cannot be avoided for real-valued representations. Pivato (2014) provided AU-type representations using nonstandard real numbers, avoiding both continuity and Archimedean axioms. For empirical and conceptual purposes, this approach is preferable to the use of standard real numbers. However, most researchers are not familiar with nonstandard real numbers.

CRediT authorship contribution statement

Chen Li: Conceptualization, Formal analysis, Investigation, Methodology, Writing – original draft, Writing – review & editing. **Peter P. Wakker:** Conceptualization, Formal analysis, Investigation, Methodology, Writing – original draft, Writing – review & editing.

Declaration of competing interest

None.

Data availability

No data was used for the research described in the article.

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Appendix A. Proofs

Our proofs are based on Theorem 7 of Kothiyal et al. (2014). Their domain of preference consisted of all finite sequences $x^* = (x_1, \ldots, x_n)$ of any length n, and a preference relation $\succcurlyeq^*$ on it. Because there was no upper bound to the length n, their domain can be called half-infinite-dimensional. It can be taken as isomorphic to our space of periodic sequences (details are provided in the proof of Observation 1). Their preference conditions are the natural analogs of our p-preference conditions. Their Theorem 7 showed that AU holds if and only if preferences satisfy (1) weak ordering; (2) Archimedeanity; (3) fairness (called symmetry); (4) independence; and one further condition called *replication equivalence*, i.e.: $x^* \sim^* mx^*$ for all finite sequences x^*. Here mx^* again denotes the m-fold, finite, periodic replication of x^*.

Proof of Observation 1. It readily follows that Statement (i) implies Statement (ii). P-independence follows because common terms in the AU summations cancel. We next assume Statement (ii) and derive Statement (i). We consider the domain of Kothiyal et al. (2014) defined above. We define a preference relation $\succcurlyeq^*$ on this domain by $(x_1, \ldots, x_n) \succcurlyeq^* (y_1, \ldots, y_m)$ if $[x_1,..,x_n] \succcurlyeq [y_1,..,y_m]$. Given weak ordering, this definition implies replication equivalence of $\succcurlyeq^*$. Kothiyal et al.'s domain is isomorphic to our domain of periodic sequences if we identify all periodic extensions mx^* with x^* in their domain. All conditions of their Theorem 7 follow, implying an AU representation of $\succcurlyeq^*$ and, accordingly, one of $\succcurlyeq$ on F^p.

Proof of Theorem 3. We first assume Statement (i) and derive Statement (ii). The first four conditions, weak ordering, p-

[11] So do the Archimedean axiom and most continuity axioms ($\forall \varepsilon\ \exists \delta$).

Archimedeanity, p-fairness, and p-independence, follow directly, and also by Observation 1.

To derive v-boundedness of every act, we assume, for contradiction, that f is v-unbounded above. Then so is $U(\Gamma)$: because Γ is unbounded, Eq. (3) holds, implying $U(\delta) - U(\gamma) > n(U(\gamma) - U(\beta))$. So, $U(\Gamma)$ is indeed unbounded above. Eq. (5) implies that $limsup_{n\to\infty} AU[f_1, \ldots, f_n]$ exceeds any $U(\gamma)$ and, hence, any real number. This contradicts that $AU(f)$ is well-defined and finite. V-unboundedness below similarly leads to a contradiction and f must be v-bounded.

We next assume, for contradiction, that f is not stable. Then there exist p-streams $x \succ y$ with $[f_1, \ldots, f_n] \succ x$ for infinitely many n and $[f_1, \ldots, f_n] \prec y$ for infinitely many n. This would imply $limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} U(f_j) \geq AU(x) > AU(y) \geq liminf_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} U(f_j)$, contradicting well-definedness of $AU(f)$.

For p-denseness, assume $f \succ g$. Then $AU(f) > AU(g)$, i.e., $AU(f) - \varepsilon > AU(g) + \varepsilon$ for some $\varepsilon > 0$. Then $\exists N \in \mathbb{N} : \forall n \geq N : AU[f_1, \ldots, f_n]\rangle AU(f) - \varepsilon > AU(g) + \varepsilon > AU[g_1, \ldots, g_n]$. Because of denseness of $AU(F^p)$ (Observation 2), there are p-streams x, y such that $AU[f_1, \ldots, f_n]\rangle AU(f) - \varepsilon > AU(x) > AU(y) > AU(g) + \varepsilon > AU[g_1, \ldots, g_n]$ for all $n \geq N$. P-denseness holds and Statement (ii) has been proved.

We next assume Statement (ii) and derive Statement (i). We have AU on F^p by Observation 1, providing U.

Assume, for contradiction, $limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} U(f_j) = \infty$. Then $U(\Gamma)$ is unbounded above, readily implying that Γ is unbounded above and so is f: a contradiction has resulted. Similarly, $liminf_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} U(f_j) = -\infty$ cannot be.

Next assume, for contradiction, $liminf_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} U(f_j) < limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} U(f_j)$. Then we can find real numbers $\mu < \nu$ in between such that $\frac{1}{n}\sum_{j=1}^{n} U(f_j) < \mu$ for infinitely many n and $\nu < \frac{1}{n}\sum_{j=1}^{n} U(f_j)$ for infinitely many n. By denseness of $AU(F^p)$ (Observation 2), we can find periodic x, y with $AU(y) < AU(x)$ between μ and ν, implying $\frac{1}{n}\sum_{j=1}^{n} U(f_j) < AU(y)$ for infinitely many n and $\frac{1}{n}\sum_{j=1}^{n} U(f_j) > AU(x)$ for infinitely many n. This contradicts stability of f.

We can conclude at this stage that $AU(f)$ is well-defined and finite for all f. We finally show that it is representing. Assume $f \succ g$. By p-denseness, we have the right-hand side of Eq. (7), which implies $AU(f) \geq AU(x) > AU(y) \geq AU(g)$, so that $AU(f) > AU(g)$. Conversely, assume $AU(f) > AU(g)$. By denseness of $AU(F^p)$ (Observation 2), we can find $\varepsilon > 0$ and p-streams x, y and m, n such that $AU([f_1, \ldots, f_n]) > AU(f) - \varepsilon > AU(x) > AU(y) > AU(g) + \varepsilon > AU([g_1, \ldots, g_m])$. There exists N such that for all $n > N$: $\frac{1}{n}\sum_{j=1}^{n} U(f_j) > AU(f) - \varepsilon > AU(x) > AU(y) > AU(g) + \varepsilon > \frac{1}{n}\sum_{j=1}^{n} U(g_j)$. By p-denseness, $f \succ g$. We have shown: $f \succ g \Leftrightarrow AU(f) > AU(g)$, i.e., AU represents the preference relation.

We, finally, establish that U is an interval scale. It is obvious that we can add any constant and multiply by any positive constant. It readily follows from Observation 4 that we do not have more liberty.

We make two further comments: for $U(\Gamma)$ to be unbounded above, it does not suffice to require that there is no maximal outcome, as for instance with $U(\Gamma) = (0,1)$. In that case, there can still exist a maximal stream f, for instance for $U(f_j) = 1 - 1/j$, but no maximal outcome. Further, for p-denseness, it is not enough to require that for every $f \succ g$ there exists a p-stream x with $f \succ x \succ g$. Then "infinitesimal" strict preferences could exist between different streams with the same AU value.

Proof of Observation 6. Continuity of U immediately implies continuity of $\succcurlyeq$. Continuity of $\succcurlyeq$ implies continuity of U by Wakker (1988 Theorem 3.1).

Appendix B. independence of the Axioms

We show that the axioms in (ii) in Theorem 3 are logically independent, showing also that none could have been omitted. In all cases below exactly one axiom is violated so that there is no AU representation. In all cases, if a functional V is defined, it represents preferences. If defined through generators, it is always well-defined, i.e., independent of which generator is chosen for p-streams. That is, it then satisfies Kothiyal et al.'s (2014) replication-invariance.

1. [weak ordering] We take preferences incomplete. Assume that $f \sim f$ for all f, and that any permutation of the x_is in any p-stream $[x_1, \ldots, x_n]$ leaves the stream indifferent. There are no other preferences. Then p-fairness holds. All axioms hold except Axiom 1.
2. [p-Archimedeanity]. We define a lexicographic representation using two different "disjoint" AU representations. Assume $\Gamma = \mathbb{R}$. $U_1 = 0$ on $(-\infty, 0]$ and $U_1(\alpha) = \alpha$ on $[0,\infty)$. $U_2(\alpha) = \alpha$ on $(-\infty, 0]$ and $U_2 = 0$ on $[0,\infty)$. AU_i denotes average with respect to U_i, $i = 1, 2$.

   F contains all bounded streams with both averages AU_1 and AU_2 well-defined. The preference relation is represented lexicographically by (AU_1, AU_2). That is, $f \succ g$ if $AU_1(f) > AU_1(g)$ or $AU_1(f) = AU_1(g)$ and $AU_2(f) > AU_2(g)$. Further, $f \sim g$ if $AU_1(f) = AU_1(g)$ and $AU_2(f) = AU_2(g)$. Axioms 1,3,4,5,6 hold trivially. For p-denseness, assume $f \succ g$. If $AU_1(f) > AU_1(g)$, then p-denseness follows from p-denseness of AU_1. Next assume $AU_1(f) = AU_1(g)$ and $AU_2(f) > AU_2(g)$. For odd j, we define $x_j = y_j = 2AU_1(f) = 2AU_1(g) \geq 0$. Next take $AU_2(f) > \alpha > \beta > AU_2(g)$. For even j, $x_j = 2\alpha$ and $y_j = 2\beta$. Now x and y are periodic and $AU_1(x) = AU_1(y) = AU_1(f) = AU_1(g)$, $AU_2(f) > AU_2(x) > AU_2(y) > AU_2(g)$. This shows that Axiom 7, p-denseness, also holds. All axioms hold except Axiom 2.
3. [*p-fairness*]. We take "nonsymmetric" discounted utility. Assume $\Gamma = \mathbb{R}$, $F = F^p$, U is the identity, and $V(x) = \sum_{j=1}^{\infty}\delta^j x_j$ with $0 < \delta < 1$ (discounted value). All axioms hold except Axiom 3.
4. [p-independence] We construct a symmetric representation with sufficiently nonlinear interactions to violate separability. Assume $\Gamma = [0,\infty)$, $F = F^p$. $V([x_1, \ldots x_n]) = \sum_{i,j=1}^{n}\frac{x_i^2 x_j}{n^2}$. We show that p-independence is violated: $V([0,1,1]) = 4/9 = V([0,0,\sqrt[3]{4}])$ but $V([1,1,1]) = 1 < 1.012 = V([1,0,\sqrt[3]{4}])$. All axioms hold except Axiom 4.

5. [*v-bounded*]. We add one unbounded stream. Assume $\Gamma = \mathbb{R}$, $U(\alpha) = \alpha$, and F contains all streams with well-defined finite AU value and one extra element: $f_j = j$ for all j. f is strictly preferred to all other streams, and AU represents preferences between all other streams. All axioms hold except Axiom 5.
6. [*stability*]. We add one nonstable stream. Assume $\Gamma = \mathbb{R}$, $U(\alpha) = \alpha$, and F contains all streams with well-defined finite AU value and one extra bounded element g with $liminf_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} g_j = 0$ and $limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} U(g_j) = 1$. We define $V(g) = 0.5$ and $V(f) = AU(f)$ for all other f. All axioms hold except Axiom 6.
7. [p-denseness] We add one infinitesimal preference difference. Assume $\Gamma = \mathbb{R}$, $U(\alpha) = \alpha$, and F contains all streams with well-defined finite AU value. We take one bounded g that is nonperiodic. AU represents preference with one exception: if $AU(g) = AU(f)$ and $g \neq f$ then $g \succ f$. All axioms hold except Axiom 7.

References

Aczél, János, 1966. Lectures on Functional Equations and Their Applications. Academic Press, New York.
Adler, Matthew D., 2019. Measuring Social Welfare: An Introduction. Oxford University Press, New York.
Arrow, Kenneth J., 1971. Essays in the Theory of Risk Bearing. North-Holland, Amsterdam.
Asheim, Geir B., 2010. Intergenerational Equity. Annu. Rev. Econom. 2, 197–222.
Asheim, Geir B., Zuber, Stéphane, 2014. Escaping the repugnant conclusion: rank-discounted utilitarianism with variable population. Theoretic. Econ. 9, 629–650.
Basu, Kaushik, Mitra, Tapan, 2007. Utilitarianism for infinite utility streams: a new welfare criterion and its axiomatic characterization. J. Econ. Theory 133, 350–373.
Baucells, Manel, Villasís, Antonio, 2015. Equal tails: a simple method to elicit utility under violations of expected utility. Decis. Anal. 12, 190–204.
Blackorby, Charles, Bossert, Walter, Donaldson, David, 2005. Population Issues in Social Choice Theory, Welfare Economics and Ethics. Cambridge University Press, Cambridge, UK.
Blackorby, Charles, Davidson, Russell, Donaldson, David, 1977. A homiletic exposition of the expected utility hypothesis. Economica 44, 351–358.
Bossert, Walter, Sprumont, Yves, Suzumura, Kotaro, 2007. Ordering infinite utility streams. J. Econ. Theory 135, 579–589.
Chambers, Christopher P., Echenique, Federico, Lambert, Nicolas S., 2021. Recovering preferences from finite data. Econometrica 89, 1633–1664.
Chew, Soo Hong, Karni, Edi, 1994. Choquet expected utility with a finite state space: commutativity and Act-independence. J. Econ. Theory 62, 469–479.
da Volterra, Gian Francesco Lottini (1574) "Avvedimenti Civili".
Debreu, Gérard, 1960. Topological methods in cardinal utility theory, 1960, eds.. In: Arrow, Kenneth J., Karlin, Samuel, Suppes, Patrick (Eds.), Mathematical Methods in the Social Sciences. Stanford University Press, Stanford, CA, pp. 16–26.
Diamond, Peter A., 1965. The evaluation of infinite utility streams. Econometrica 33, 170–177.
Drèze, Jacques H., 1987. Essays on Economic Decision Under Uncertainty.". Cambridge University Press, Cambridge.
Elster, John, 1986. Introduction. In: Elster, John (Ed.), Rational Choice. New York University Press, New York, pp. 1–33.
Fishburn, Peter C, Ward, Edwards, 1997. Discount-neutral utility models for denumerable time streams. Theory Decis. 43, 139–166.
Gale, David, 1967. An optimal development in a multi-sector economy. Rev. Econ. Stud. 34, 1–18.
Gorman, William M., 1968. The structure of utility functions. Rev. Econ. Stud. 35, 367–390.
Gravel, Nicolas, Marchant, Thierry, Sen, Arunava, 2012. Uniform expected utility criteria for decision making under ignorance or objective ambiguity. J. Math. Psychol. 56, 297–315.
Gravel, Nicolas, Marchant, Thierry, Sen, Arunava, 2018. Conditional expected utility criteria for decision making under ignorance or objective ambiguity. J. Math. Econ. 78, 79–95.
Gul, Faruk, 1992. Savage's theorem with a finite number of states. J. Econ. Theory 57, 99–110 ("Erratum," 1993, Journal of Economic Theory 61, 184.).
Harvey, Charles M., 1986. Value functions for infinite-period planning. Manage. Sci. 32, 1123–1139.
Howson, Colin, 2014. Finite additivity, another lottery paradox and Conditionalisation. Synthese 191, 989–1012.
Jaffray, Jean-Yves, 1974. On the extension of additive utilities to infinite sets. J. Math. Psychol. 11, 431–452.
Jevons, W.Stanley, 1871. The Theory of Political Economy, 5th edn. Kelley and MacMillan, New York. 1957; other edn. Penguin, 1970.
Jonsson, Adam, Voorneveld, Mark, 2018. The limit of discounted utilitarianism. Theor. Econ. 13, 19–37.
Khan, Urmee, Stinchcombe, Maxwell B., 2018. Planning for the long run: programming with patient, Pareto responsive preferences. J. Econ. Theory 176, 444–478.
Khan, M.Ali, Uyanik, Metin, 2021. Topological connectedness and behavioral assumptions on preferences: a two-way relationship. Econ. Theory 71, 411–460.
Kothiyal, Amit, Spinu, Vitalie, Wakker, Peter P., 2014. Average utility maximization: a preference foundation. Oper. Res. 62, 207–218.
Kraitchik Maurice, B, 1953. Mathematical Recreations. Dover, New York, NY.
Krantz, David H., Luce, Duncan R., Suppes, Patrick, Tversky, Amos, 1971. Foundations of Measurement, Vol. I (Additive and Polynomial Representations), 2nd edn. Dover Publications, New York. Academic Press, New York, 2007.
Laplace, Pierre Simon de, 1796. Essai Philosophique sur les Probabilités, Paris, 1825. Translated Into English as A Philosophical Essay on Probabilities, 5th edn. Dover Publications, New York, p. 1951.
Lauwers, Luc, 1998. Intertemporal objective functions: strong Pareto versus anonymity. Math. Soc. Sci. 35, 37–55.
Lauwers, Luc, 2012. Intergenerational equity, efficiency, and constructability. Econ. Theory 49, 227–242.
Luce, R.Duncan, Krantz, David H., Suppes, Patrick, Tversky, Amos, 1990. Foundations of Measurement, Vol. III. (Representation, Axiomatization, and Invariance). Academic Press, New York.
Marinacci, Massimo, 1998. An axiomatic approach to complete patience and time invariance. J. Econ. Theory 83, 105–144.
Mitra, Tapan, Basu, Kaushik, 2007. On the existence of Paretian social welfare quasi-orderings for infinite utility streams with extended anonymity. In: Roemer, John, Suzumura, Kotaro (Eds.), Intergenerational Equity and Sustainability. Palgrave, London, 2007.
Mosteller, Frederick, Nogee, Philip, 1951. An experimental measurement of utility. J. Politic. Econ. 59, 371–404.
Münnich, Àkos, Maksa, Gyula, Mokken, Robert J., 2000. n-Variable Bisection. J. Math. Psychol. 44, 569–581.
Nalebuff, Barry J., 1989. The other Person's envelope is always greener. J. Econ. Perspect. 3, 171–181.
Neyman, Abraham, 2023. Additive valuations of streams of payoffs that satisfy the time value of money principle: a characterization and robust optimization. Theoretic. Econ. 18, 303–340.
Peters, Hans J.M., 2015. Game Theory; A Multi-Leveled Approach, 2nd edn. Springer, Berlin.
Petri, Henrik, 2019. Asymptotic properties of welfare relations. Econ. Theory 67, 853–874.
Pfanzagl, Johann, 1968. Theory of Measurement. Physica-Verlag, Vienna.
Pigou, Arthur C., 1920. The Economics of Welfare. MacMillan, London edn. 1952.
Pivato, Marcus, 2014. Additive representation of separable preferences over infinite products. Theory Decis. 73, 31–83.
Pivato, Marcus, 2022. A characterization of Cesàro average utility. J. Econ. Theory 201, 105440.

Rébillé, Yan, 2007. Patience in some non-additive models. J. Math. Econ. 43, 749–763.
Sidgwick, Henry, 1874. The Methods of Ethics, 7th edn. MacMillan, London. 1907.
Svensson, Lars-Gunnar, 1980. Equity among generations. Econometrica 48, 1251–1256.
Thomson, William, 2001. On the axiomatic method and its recent applications to game theory and resource allocation. Soc. Choice Welfare 18, 327–386.
Wakker, Peter P., 1988. The algebraic versus the topological approach to additive representations. J. Math. Psychol. 32, 421–435.
Wakker, Peter P., 1993. Savage's axioms usually imply violation of strict stochastic dominance. Rev. Econ. Stud. 60, 487–493.
Wakker, Peter P., Zank, Horst, 1999. State dependent expected utility for Savage's state space; or: Bayesian statistics without prior probabilities. Math. Oper. Res. 24, 8–34.
Weinstein, Milton C., 1993. Time-preference studies in the health care context,". Med. Decis. Making 218–219.
Weymark, John A., 1995. Further remarks on Harsanyi's social aggregation theorem and the weak Pareto principle. Soc. Choice Welfare 12, 87–92.
Yi, Byeong-Uk, 2013. Conditionals and a two-envelope paradox. J. Philos. 110 (5), 233–257.
Zuber, Stéphane, Asheim, Geir B., 2012. Justifying social discounting: the rank-discounted utilitarian approach. J. Econ. Theory 147, 1572–1601.