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A Single-Stage Approach to Anscombe and Aumann's Expected Utility

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Anscombe and Aumann showed that if one accepts the existence of a physical randomizing device such as a roulette wheel then Savage's derivation of subjective expected utility can be considerably simplified. They, however, invoked compound gambles to define their axioms. We demonstrate that the subjective expected utility derivation can be further simplified and need not invoke compound gambles. Our simplification is obtained by closely following the steps by which probabilities and utilities are elicited.

1. INTRODUCTION

The most well-known justification for subjective expected utility theory (SEU) was provided by Savage (1954). Savage's hallmark contribution was to derive both utilities and probabilities from preferences. His axioms are appealing but the analysis is complex and requires a rich set of available acts and states of the world. Soon after publication of Savage's work, several authors attempted to simplify the derivation of SEU (Rubin (1949), Blackwell and Girshick (1954), Chernoff (1954), Anscombe and Aumann (1963), Pratt, Raiffa and Schlaifer (1964)). The simplifications were obtained by introducing an objective randomizing device such as a roulette wheel. These alternative derivations utilized a two-stage setup where an act yields a probability distribution under each state of the world. In contrast, Savage's act yields a prize (degenerate lottery) under each state of the world. This added complexity of the two-stage setup paid dividends in the simplification of the state space and of the axioms and proofs.

Anscombe and Aumann's (1963) (AA) approach for deriving SEU has been found to be the most attractive. They adapt the independence axiom of von Neumann and Morgenstern (1944) to the two-stage setup by means of a "reversal of order in compound lotteries" assumption. They permit finite state spaces, thus avoiding several mathematical complications (Stinchcombe (1994)). The AA approach attained a celebrity status because the axioms are highly transparent and the proofs are simple (Kreps (1988)). Subsequently, many authors have fruitfully employed the AA setup in refinements and modifications of the SEU model (Fishburn (1970, 1982), Hazen (1987, 1989), Schmeidler (1989), Karni (1993), Nau (1993), Eichberger and Kelsey (1993), Machina and Schmeidler (1995), Lo (1996)).

The aim of this paper is to further simplify the AA approach. Like AA, we use a randomizing device for calibrating subjective probabilities and utilities. One distinction between our approach and that of AA is that, in AA's formulation, under each uncertain event the outcome can be any lottery, whereas in our approach we only need to assume that the outcome is a prize, i.e. a degenerate lottery. Thus we avoid the two-stage setup. This strategy of employing a single-stage approach rather than the traditional two-stage setup has been used by Sarin and Wakker (1992) to simplify Schmeidler's (1989) derivation of Choquet-expected utility. Complications that are introduced by two-stage setups have been described by Loomes and Sugden (1986), Segal (1990), and Luce and von Winterfeldt (1994).

A second distinction between our approach and that of AA is that in our setup we need not invoke all assignments of prizes to events. The two distinctions imply that we need fewer structural assumptions and obtain greater flexibility in modelling. Arguments for the need for structural simplicity have been given by Aumann (1962, 1971), Krantz, Luce, Suppes, and Tversky (1971), Suppes, Krantz, Luce and Tversky (1989), Fishburn (1967, 1976), and Nau (1992).

The strategy in our axiomatization is to closely follow the steps of elicitation commonly employed in decision analysis. Consistency conditions are imposed that serve to prevent contradictions in the elicitations. The resulting axioms are highly transparent and the proofs are elementary. We hope that the obtained simplicity is considered a virtue.

The consistency conditions are provided in Section 2. In Section 3, a formal presentation of our axioms and characterization theorem is given. Section 4 describes some examples that illustrate the simplicity of our model. Section 5 presents an alternative characterization through a dominance axiom; the dominance axiom resembles stochastic dominance. Section 6 provides a conclusion. Proofs are given in the appendix.

2. CONSISTENCY CONDITIONS

The problem addressed here is the evaluation of an act $(H_1, x_1; \ldots; H_n, x_n)$ yielding a prize x_j if event H_j occurs, where probabilities of the events H_j are not given. The events H_j are disjoint and exhaustive. AA gave an example where the H events refer to the outcome of a horse race. Their terminology has been generally accepted, therefore we call these events "horse events." We do not impose restrictions on prizes; they could be quantitive variables such as money or could be qualitative such as good or bad health.

In the evaluation of a gamble two imputs are required. One is the probability of events and the other is the utility of prizes. Two elicitation assumptions, one dealing with the elicitation of utilities and the other with the elicitation of probabilities, and one valuation assumption give us the desired SEU of a gamble.

We follow AA in using a randomizing device such as a roulette wheel for calibrating utilities and subjective probabilities. Physical ("known," "objective") probabilities are generated by using the roulette wheel. Thus, any desired probability distribution over prizes can be specified in terms of a lottery contingent on the outcomes of the roulette wheel. Following AA, we assume:

Assumption 2.1 (*Elicitation*). Preferences over roulette wheel lotteries are represented by ("von Neumann–Morgenstern") expected utility.

Assumption 2.1 is utilized to elicit utilities over prizes by standard procedures (Mosteller and Nogee (1951), Raiffa (1968), Farquhar (1984)). In spite of the expected utility assumption for lotteries it is by no means clear that preferences over uncertain acts will satisfy SEU (Ellsberg (1961), Schmeidler (1989)). Keynes (1921) and Knight (1921) emphasized the distinction between risk and uncertainty. Ramsey (1931), de Finetti (1937), and Savage (1954) argued that subjective probabilities can be assigned to uncertain events. The first step to obtain an SEU representation for acts is to show that beliefs about uncertain events (horse events) can be quantified by probabilities.

In the standard procedure to calibrate the probability of an event H, a favourable prize x^* and an unfavourable prize x_* are fixed. Then a roulette-wheel probability p is elicited such that the decision maker is indifferent between the act $(H, x^*; \textit{not-H}, x_*)$ and the lottery $(p, x^*; 1-p, x_*)$. This indifference is displayed in Figure 1.

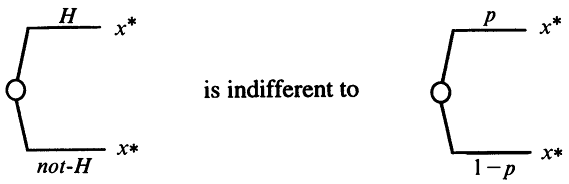

FIGURE 1

The method for eliciting probabilities just described is based on a comparison of likelihoods of different events through bets on events. Events E and E' are equally likely if one equally prefers betting on E to betting on E', i.e. $(E, x^*; \textit{not-E}, x_*)$ is indifferent to $(E', x^*; \textit{not-E}', x_*)$. In this case we write $E \sim_l E'$. This method for calibrating probabilities has been well understood throughout history (Borel (1939, Sections 39 and 48)).

A consistency check for subjective probabilities will require that if the probability of H is assessed to be p, the probability of a disjoint event H' is assessed to be q, and in a cross-checking the probability of the union $H \cup H'$ is assessed to be r, then r must be equal to $p+q$. This additivity of probability is ensured by the following assumption on likelihood comparisons through bets on events:

Assumption 2.2 (*Elicitation*). For all disjoint horse events H, H' there exist disjoint roulette events R, R' such that $H \sim_l R$, $H' \sim_l R'$, and $H \cup H' \sim_l R \cup R'$.

While Assumption 2.2 may seem self-evident, we note that, as early as 1949, Shackle had argued for nonadditive probabilities. Ellsberg (1961) provided some ingenious examples where Assumption 2.2 is violated. Recently, many authors (for example, Schmeidler (1989), Gilboa (1987)), have developed alternative models for decision under uncertainty that relax Assumption 2.2. Assumptions 2.1 and 2.2 guarantee that utilities are assigned to prizes and that beliefs can be quantified by additive probabilities. We now turn to the valuation of acts, for which it must be specified how probabilities and utilities are aggregated into an overall valuation. At this stage it is still possible that the decision maker uses methods of valuation that deviate from SEU. A way to elicit the value of an act $(H_1, x_1; \ldots; H_n, x_n)$ is as follows. First one elicits the probabilities $p_1, \ldots, p_n$ of the events $H_1, \ldots, H_n$ using the roulette wheel. By additivity, assured by Assumption 2.2, $p_1, \ldots, p_n$ sum to one. Therefore we can construct a "matching" lottery $(R_1, x_1; \ldots; R_n, x_n)$, where the events $R_1, \ldots, R_n$ are roulette wheel events with probabilities $p_1, \ldots, p_n$ respectively. By Assumption 2.1, the roulette wheel lottery is valued by the *expected utility* $\sum_{j=1}^{n} p_j U(x_j)$. For consistency we would want the value of the act $(H_1, x_1; \ldots; H_n, x_n)$ to be the same. This consistency is ensured by the following condition.

Assumption 2.3 (*Valuation*). An act $(H_1, x_1; \ldots; H_n, x_n)$ is indifferent to a lottery $(R_1, x_1; \ldots; R_n, x_n)$ whenever $H_1 \sim_l R_1, \ldots, H_n \sim_l R_n$.

Assumption 2.3 is based on the general principle that two gambles be judged indifferent if each prize is equally likely under the two. This method of evaluation of an uncertain act by matching it with a lottery with the same probability distribution has been widely used in decision analysis (Raiffa (1968, Section 5.3 and page 109/110), Schlaifer (1969, Section 4.4.5)). Although the conditions of this section are guided by the elicitation process and serve to avoid contradictions in the assessments, they are easily reformulated as preference conditions. This is obtained mainly by replacing the equal-likelihood conditions in Assumptions 2.2 and 2.3 by their definitions in terms of bets on events. It is straightforward to observe that Assumption 2.2 implies additivity of probability and Assumption 2.3 implies an SEU valuation for all acts. In this manner the assumptions of this section, along with some common assumptions that are described in the next section, provide an elementary characterization of SEU.

3. A FORMAL PRESENTATION

Our setup involves two basic elements. One is the horse event H and the other is the roulette event R. The set of horse events, denoted by $\mathscr{H}$, may be viewed as containing subsets of an underlying horse-state space S (finite or infinite). Some richness conditions concerning $\mathscr{H}$ are specified in Assumption 3.1. The set of roulette events, $\mathscr{R}$, contains all the subintervals of $[0, 1)$;[1] to each interval R a probability is assigned that corresponds to the length of the interval. Our notation is as follows.

X	An arbitrary set of *prizes*;[2]
$(H_1, x_1; \ldots; H_n, x_n)$	An *act* ("pure horse lottery") yielding prize x_j in event H_j;
$\mathscr{A}$	The set of all acts;
$(R_1, x_1; \ldots; R_n, x_n) = (p_1, x_1; \ldots; p_n, x_n)$[3]	A ("pure roulette") *lottery* yielding x_j in roulette event R_j with probability p_j $(1 \leq n < \infty, \sum_{j=1}^{n} p_j = 1)$;
$\mathscr{L}$	The set of all (roulette) lotteries;
$G = (E_1, x_1; \ldots; E_n, x_n)$	A *gamble*, i.e. either an act or a lottery;
$\mathscr{G} = \mathscr{A} \cup \mathscr{L}$	The set of all gambles;
$\succsim$	The decision maker's preference relation on $\mathscr{G}$.

The notation $\succ$, $\sim$, $\precsim$, and $\prec$ is as usual. Acts are functions from the underlying horse state space S to X. Thus the events $H_1, \ldots, H_n$ partition the horse state space. Lotteries are functions from $[0, 1)$ to X, and the roulette events $R_1, \ldots, R_n$ partition the roulette wheel state space $[0, 1)$. Prizes are identified both with degenerate lotteries and with constant acts.[4] Thus preferences over prizes are generated by preferences over degenerate lotteries, and preferences over constant acts necessarily coincide with those preferences. To avoid triviality we assume throughout that two non-indifferent prizes $x^* \succ x_*$ are

1. The only reason for taking $[0, 1)$ instead of $[0, 1]$ is that the notation of partitions into intervals is now somewhat simpler; the intervals can all be left-closed and right-open.

2 X can be finite as well as infinite.

3. The identification of a lottery with the induced probability distribution over prizes will be justified in the sequel.

4. It is sometimes convenient to assume one joint underlying state space $S \times [0, 1)$ that specifies both the uncertainties regarding horses and regarding the roulette wheel. Then a roulette wheel event R is related to the subset $S \times R$ of the joint state space and a horse event H to the subset $H \times [0, 1)$ of the joint state space. In this case, degenerate lotteries and constant acts are formally identical.

available. These two prizes are fixed throughout and are used to obtain likelihood comparisons.

The set $\mathscr{L}$ of roulette lotteries consists of all roulette lotteries $(R_1, x_1; \ldots; R_n, x_n)$ for prizes $x_1, \ldots, x_n$ and subintervals $R_1, \ldots, R_n$ partitioning $[0, 1)$. Throughout we assume, without further mention, that all lotteries that generate the same probability distribution over prizes are indifferent. Therefore we often denote lotteries simply by the generated probability distributions. This is the common approach in decision under risk. Note that all simple probability distributions over prizes can be generated by the roulette wheel.

We do not assume that $\mathscr{A}$ contains all assignments of prizes to horse events. We describe our assumptions regarding horse events and acts later. We will often use two-outcome gambles $(E, x^*; \textit{not-}E, x_*)$ in our analysis, hence we introduce the simplifying notation (E, x^*) for such gambles; similarly, (p, x^*) denotes $(p, x^*; 1-p, x_*)$.

Assumption 3.1 (*Domain*).[5]

$\mathscr{H}$ contains:

$H_1, \ldots, H_n$ for each act $(H_1, x_1; \ldots; H_n, x_n)$ with $x_i \neq x_j$ whenever $i \neq j$;

the union of all of its pairs of elements.

$\mathscr{A}$ contains the act (H, x^*) for each event $H \in \mathscr{H}$.

Equivalently, we could have first defined the collection $\mathscr{H}$ of events, and then define $\mathscr{A}$ as a subset of the set of functions from S to X that are "measurable" with respect to $\mathscr{H}$. That is the most natural approach for Savage's (1954) model and other models where all functions from S to X are incorporated as acts. In our setup, where the set $\mathscr{A}$ can be fairly general, we preferred the setup where $\mathscr{A}$ is defined first and $\mathscr{H}$ is derived from $\mathscr{A}$.

Now we turn to preference axioms that characterize SEU. Our axioms concern preferences over acts in $\mathscr{A}$, preferences over lotteries in $\mathscr{L}$ and preferences where an act is compared to a lottery. Of course, both $\mathscr{A}$ and $\mathscr{L}$ are contained in $\mathscr{G}$ so, to avoid repetition, our axioms are formulated in terms of $\mathscr{G}$.

Axiom 3.2 (*Weak Ordering*). The preference relation $\succeq$ on $\mathscr{G}$ is complete and transitive.

The following axiom imposes a dominance condition. It implies that the likelihood of each event is between the likelihoods of the universal event and the impossible event. Alternatively, it can be interpreted as a monotonicity condition with respect to prizes, saying that replacing x_* by x^* is desirable.

Axiom 3.3 (*Monotonicity*). $x^* \succeq (H, x^*) \succeq x_*$ for all $H \in \mathscr{H}$.

For lotteries the classical independence and continuity axioms are imposed; we use Jensen's (1967) versions.

Axiom 3.4 (*Independence axiom for lotteries*). For lotteries $L, L', Q \in \mathscr{L}$, if $L \succ L'$ and $0 < \lambda \leqq 1$, then $\lambda L + (1-\lambda)Q \succ \lambda L' + (1-\lambda)Q$.[6]

5. Observation 1 in the appendix demonstrates that our assumptions imply that $\mathscr{H}$ is an "algebra."

6. Here $\lambda L + (1-\lambda)L'$ assigns to each prize x the probability $\lambda L(x) + (1-\lambda)L'(x)$, where $L(x)$ and $L'(x)$ are the probabilities assigned to x by L and L', respectively.

Axiom 3.5 (*Jensen Continuity for Lotteries*). For lotteries L, L', L'', if $L \succ L' \succ L''$, then there exist $0<\lambda<1$ and $0<\mu<1$ such that $\lambda L+(1-\lambda)L'' \succ L' \succ \mu L+(1-\mu)L''$.

The next axioms reformulate Axioms 2.2 and 2.3 in terms of preference conditions.

Axiom 3.6 (*Additivity*). For all disjoint horse events H, H' there exist disjoint roulette events R, R' such that $(H, x^*) \sim (R, x^*)$, $(H', x^*) \sim (R', x^*)$, and $(H \cup H', x^*) \sim (R \cup R', x^*)$.

Axiom 3.7 (*Probabilistic Beliefs*). Act $(H_1, x_1; \ldots; H_n, x_n)$ is indifferent to lottery $(R_1, x_1; \ldots; R_n, x_n)$ whenever $(H_i, x^*) \sim (R_i, x^*)$ for all i.

We now state our main result.

Theorem 3.8. *Under the Domain Assumption* 3.1, *the following two statements are equivalent*:

(i) SEU *holds*; *i.e., there exists a utility function* $U: X \rightarrow \mathbb{R}$ *and the probability* P *can be extended to the horse events, such that preferences over gambles are represented by the function*

$$(E_1, x_1; \ldots; E_n, x_n) \mapsto \sum_{j=1}^{n} P(E_j)U(x_j).$$

(ii) $\succeq$ *satisfies Axioms* 3.2 (*weak ordering*), 3.3 (*monotonicity*), 3.4 (*independence axiom*), 3.5 (*Jensen continuity*), 3.6 (*additivity*), *and* 3.7 (*probabilistic beliefs*).

4. EXAMPLES

This section presents two examples to illustrate the flexibility of our approach. In both examples, horses participate in a race, exactly one horse will win, and it is unknown which horse will win. The first example describes a simple single-stage approach that is compatible with our assumptions. The second example further illustrates the increased flexibility that is possible in our approach, requiring fewer hypothetical assumptions.

Example 4.1. There are two horses s, t. Acts are mappings from $\{s, t\}$ to X, i.e. the prize resulting from an act depends on the horse that will win the race. Horse events are subsets of $\{s, t\}$. Roulette lotteries are generated by the random drawing of a number from $[0, 1)$, as in the previous sections. Obviously, a subjective probability p for s means that an act assigning a prize x to s and a prize y to t is indifferent to a roulette lottery assigning x to the interval $[0, p)$ and y to $[p, 1)$. Note that we did not make independence assumptions, or other assumptions, about joint probability distributions of horses and numbers from $[0, 1)$. Simply, such assumptions are not needed in our setup.

Example 4.2. Assume three horses, h_1, h_2, h_3. Suppose the decision maker considers three acts: Stake \$1 on horse 1, stake \$1 on horse 2, or not bet at all. Further suppose that a bet on horse 1 yields \$10 if horse 1 wins. Then the net profit, in dollars, of betting on horse 1 is 9 if it wins and -1 if some other horse wins. Similarly, a bet on horse 2 yields \$2 if horse 2 wins. Denoting acts by the corresponding net profit vectors, the three acts considered by the decision maker are:

$$(9, -1, -1), \quad (-1, 1, -1), \quad \text{and} \quad (0, 0, 0).$$

In our model we assume all "pure" lotteries over the set of prizes $X = \{9, 1, 0, -1\}$, and from these the utilities of prizes result by standard procedures. Let us take $x^* = 1$ and $x_* = -1$ as the prizes for calibrating the probabilities of horse events. The three "imaginary" acts used for probability calibration are $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, 1)$. Our axiomatization, requiring availability of all acts $(H, x^*; H^c, x_*)$ for any horse event H, requires three more imaginary acts, i.e. the acts $(-1, 1, 1)$, $(1, -1, 1)$, $(1, 1, -1)$.[7] In elicitations, these acts can be used to verify additivity of probability and can thus provide cross-checkings for elicited probabilities.

With the pure lotteries over the prizes and the six imaginary acts just described, the domain of our theory is complete and the characterization Theorems 3.8 (and 5.3) can already be invoked. Obviously, if considered useful, one has the possibility of adding more imaginary acts. One may, for instance, add the three imaginary acts $(9, 0, 0)$, $(0, 9, 0)$, $(0, 0, 9)$ to provide alternative calibrations of probability. Then still our characterization theorems apply and give a foundation to SEU. In short, there is considerable flexibility concerning domain.

AA invoke more imaginary choice alternatives in this example, with the displayed three actual acts. Like us, their model invokes the entire set $\mathscr{L}$ of lotteries over $X = \{9, 1, 0, -1\}$. Their model requires all $3^4 = 81$ assignments of prizes to $\{h_1, h_2, h_3\}$, which adds 78 imaginary acts, including all, unrealistic, acts that assign net wins to all horses. But then (both in its original version and in its modern version) the model also requires all assignments of lotteries-over-X to $\{h_1, h_2, h_3\}$. In the original version, the domain was extended further by also including all lotteries over assignments as just described.

5. A DOMINANCE-CHARACTERIZATION

Some may judge that the axiomatization given before is not satisfactory because the axioms are too close to the representation that they seek to characterize. Thus Axiom 3.6 may simply seem like a direct reformulation of additivity of probability. Note, however, that the axioms given before satisfy all the requirements of a characterization, i.e. they formulate conditions entirely in terms of the empirical primitive which is the preference relation.

In this section we provide an alternative characterization, that shares all the structural advantages of the analysis in the previous sections. The present characterization resembles the one of Section 3 but is based on an appeal to the intuition of stochastic dominance rather than additivity and probabilistic beliefs.

Minimal prizes of gambles play a special role here, hence we use subscripts 0 to denote them. We rank-order prizes of gambles; i.e. a typical gamble is now denoted by $(E_0, x_0; E_1, x_1; \ldots; E_n, x_n)$ where it is assumed that $x_0 \preceq x_1 \preceq \cdots \preceq x_n$. For two-outcome gambles $(\textit{not-}E, x_*; E, x^*)$ and $(1-p, x_*; p, x^*)$ we maintain the abbreviated notation (E, x^*) and (p, x^*). Now we turn to our new condition that extends the well-known idea of stochastic dominance. This condition is an alternative to the extension of stochastic dominance used by Sarin and Wakker (1992) to characterize a nonexpected utility model, "Choquet-expected utility." They use a single-stage approach similar to the present paper, and describe the axioms that should be added to their Choquet-expected utility model to characterize SEU. Their axioms are, however, more complicated than the ones presented in this paper.

Consider two lotteries $(p_0, x_0; p_1, x_1; \ldots; p_n, x_n)$ and $(q_0, x_0; q_1, x_1; \ldots; q_n, x_n)$, and assume that $p_1 \geqq q_1, \ldots, p_n \geqq q_n$. In other words, each non-minimal prize is more likely in

7. Besides the trivial acts $(1, 1, 1)$ and $(-1, -1, -1)$.

the first lottery. This implies that the first lottery stochastically dominates the second. A generally accepted condition for rational choice is that the stochastically dominating lottery be preferred. The expected utility model as implied by the von Neumann–Morgenstern axioms satisfies stochastic dominance. Strict stochastic dominance adds the requirement that the preference between the above two lotteries be strict if in addition, for some j, $x_j \succ x_0$ and $p_j > q_j$. We extend the principle of stochastic dominance to general gambles. The preference $(H, x^*) \succsim (R, x^*)$ in the next condition can be interpreted as saying that H is at least as likely as R; we then write $H \succsim_l R$. Obviously, $\sim_l$ as defined before is the symmetric part, and the asymmetric part $\succ_l$ is defined as usual.

Axiom 5.1 (*Event-Dominance*). $(H_0, x_0; H_1, x_1; \ldots; H_n, x_n) \succsim (R_0, x_0; R_1, x_1, \ldots; R_n, x_n)$ whenever $(H_1, x^*) \succsim (R_1, x^*), \ldots, (H_n, x^*) \succsim (R_n, x^*)$; further, the former preference is strict if, in addition, $x_j \succ x_0$ and $(H_j, x^*) \succ (R_j, x^*)$ for some j. Similar conditions hold when preferences are reversed ($\precsim$ instead of $\succsim$ and $\prec$ instead of $\succ$) except in $x_j \succ x_0$.

Axiom 5.1 says that, if for each non-minimal prize the associated horse event in an act is more likely than the corresponding event in the lottery, then the act must be preferred to the lottery. Note that in Axiom 5.1 it cannot be assumed *a priori* that from $H_1 \succsim_l R_1, \ldots, H_n \succsim_l R_n$ it follows that $R_0 \succsim_l H_0$. That conclusion does follow as an implication of Theorem 5.3. Next we strengthen Jensen continuity.

Axiom 5.2 (*Continuity*). For lotteries L, L', and gamble G, if $L \succ G \succ L'$, then there exist $0 < \lambda < 1$ and $0 < \mu < 1$ such that $\lambda L + (1-\lambda)L' \succ G \succ \mu L + (1-\mu)L'$.

The axiom strengthens Jensen continuity because G can be an act as well as a lottery.

Theorem 5.3. *Under the Domain Assumption* 3.1, *the following two statements are equivalent*:

(i) SEU *holds*.
(ii) $\succsim$ *satisfies Axioms* 3.2 (*weak ordering*), 3.3 (*monotonicity*), 3.4 (*independence for lotteries*), 5.1 (*event dominance*), *and* 5.2 (*continuity*).

In Theorem 5.3, event dominance and continuity replace Jensen continuity, additivity, and probabilistic beliefs of Theorem 3.8.

6. CONCLUSION

Our approach shares some similarities as well as some important differences with that of AA. We adopt their strategy of using a randomizing device (roulette wheel) to simplify the axiomatization. The important difference between our approach and that of AA is that we do not need compound gambles in our analysis. In AA's setup a horse event yields an outcome that is not a determinate prize but a probability distribution over prizes. Thus AA invoke all assignments of lotteries to horse events. In our single-stage approach, the outcomes of either a horse race or a roulette wheel spin are always final prizes (degenerate lotteries). Examples illustrating the simplicity of our approach have been given in Section 4.

The simplification of the measurement and application of expected utility obtained in this paper may be viewed as a modest contribution to a large literature that exists for deriving and justifying expected utility (for another recent justification, see Hammond (1988)). It is remarkable, however, that in spite of a long history of the study of the AA approach, no one has noticed the simpler single-stage approach. Several authors (Hazen (1987), Schmeidler (1989), Nau (1993), Machina and Schmeidler (1995) have used the two-stage setup of AA to derive new models for decision under uncertainty. We believe that the single-stage setup can lead to simplifications of these derivations. Finally, we hope that the single-stage approach will facilitate the presentation and teaching of expected utility, and the development of alternative models.

APPENDIX

Proofs and additional results

The first observation demonstrates that $\mathscr{H}$, the collection of horse events, is an "algebra." Thereafter, proofs of Theorems 3.8 and 5.3 are given.

Observation 1. *Under Assumption* 3.1 $\mathscr{H}$ *is nonempty, it is closed under finite unions, intersections, and complementation; i.e. it is an algebra.*

Proof. It was assumed in the text that all simple lotteries are contained in $\mathscr{L}$, and that the degenerate lotteries can be identified with constant acts. Hence all constant acts are contained in $\mathscr{A}$, in particular the act $(S, x^*; \varnothing, x_*)$, where S denotes the universal event. By Assumption 3.1, this implies that S and $\varnothing$ are contained in $\mathscr{H}$, and therefore $\mathscr{H}$ is nonempty. For every event H, (H, x^*) is contained in $\mathscr{H}$, i.e. $(H, x^*; H^c, x_*)$ is contained in $\mathscr{H}$. This implies that not only H, but also its complement H^c is contained in $\mathscr{H}$, so $\mathscr{H}$ is closed under complementation. As it is closed under union, it is an algebra, and is closed under finite unions and intersections. ||

Proof of Theorem 3.8. Necessity of the conditions in Statement (ii) is obvious, so we show sufficiency. For each horse event H we define the "probability" $P(H)$ as the number p such that $(H, x^*) \sim (p, x^*)$. Existence of such a number p follows from Axiom 3.6 (take $H' = H^c$) and uniqueness follows from stochastic dominance on lotteries, implied by expected utility.there (Jensen (1967) and Axioms 3.2, 3.4, 3.5).

To show additivity of probability, assume that H and H' are disjoint horse events and take R and R' as in Axiom 3.6. Set $P(H) = P(R) = p$, $P(H') = P(R') = q$. Now $(H \cup H', x^*) \sim (R \cup R', x^*)$ implies that $P(H \cup H') = p + q$.

To value an act $(H_1, x_1; \ldots; H_n, x_n)$, define inductively, by Axiom 3.6, disjoint R^*_{j-1}, R_j such that $R^*_{j-1} \sim_l (H_1 \cup \cdots \cup H_{j-1})$, $(p_j =) P(H_j) = P(R_j)$, $R^*_{j-1} \cup R_j \sim_l H_1 \cup \cdots \cup H_j$. We may assume $R_j = [p_1 + \cdots + p_{j-1}, p_1 + \cdots + p_j)$ and $R^*_{j-1} = [0, p_1 + \cdots + p_{j-1}) = R_1 \cup \cdots \cup R_{j-1}$. Now $P(H_1 \cup \cdots \cup H_n) = P(R_1 \cup \cdots \cup R_n) \leqq 1$. Strict inequality is excluded because $x^* \succ (x^*, R_1 \cup \cdots \cup R_n) \sim (x^*, H_1 \cup \cdots \cup H_n) = x^*$ cannot be. Hence $R_1, \ldots, R_n$ is exhaustive. By Axiom 3.7, $(H_1, x_1; \ldots; H_n, x_n) \sim (R_1, x_1; \ldots; R_n, x_n)$ with value $\sum_{j=1}^n p_j U(x_j)$; i.e., the SEU value of the act. This completes the proof of Theorem 3.8. ||

Proof of Theorem 5.3. The proof of necessity of (ii) is immediate, so we assume (ii) and derive (i). Continuity implies Jensen continuity, hence expected utility holds for lotteries as in the proof of Theorem 3.8. Next a preparatory result is derived.

Lemma 2. *For each event* $H \in \mathscr{H}$ *there exists a unique* $p \in [0, 1]$ *such that* $(H, x^*) \sim (p, x^*)$.

Proof. By Axiom 3.3 (monotonicity), $(p, x^*) \succeq (H, x^*) \succeq (q, x^*)$ for $p = 1$ and $q = 0$. We may assume that both preferences are strict, as the other cases are trivial. Next consider any arbitrary p and q such that $(p, x^*) \succ (H, x^*) \succ (q, x^*)$.

By stochastic dominance on lotteries as implied by the expected utility representation there, $p > q$ and the same preferences hold for all $p' > p$ and $q' < q$. By Axiom 5.2 (continuity), $(p', x^*) \succ (H, x^*) \succ (q', x^*)$ for some $p' < p$ and $q' > q$: In the notation used in the definition of continuity, $p' = \lambda p + (1 - \lambda) q$ can be taken for some $0 < \lambda < 1$ and $q' = \mu p + (1 - \mu) q$ for some $0 < \mu < 1$. Apparently, the set of p such that $(p, x^*) \succ (H, x^*)$ is an interval of the form $(\sigma, 1]$ and the set of q such that $(H, x^*) \succ (q, x^*)$ is an interval of the form $[0, \tau)$. For all probabilities $\tau \leqq p \leqq \sigma$, $(p, x^*) \sim (H, x^*)$. By stochastic dominance on lotteries, $\sigma = \tau = p$ must hold; i.e. there exists a p as required and it is unique. ||

For each horse event H we define the "probability" $P(H)$ as the number p such that $(H, x^*) \sim (p, x^*)$. Existence and uniqueness of such a number p follows from Lemma 2.

The proof now proceeds in three stages. The first stage prepares for Stage 2, Lemma 2 and Stage 2 then implies Axiom 3.6, and Stage 3 derives Axiom 3.7. All events and acts used in this proof are available because of the Domain Assumption 3.1; this will not be made explicit anymore.

Stage 1. For disjoint horse events H, H', $P(H) + P(H') \leqq 1$.

To derive this stage let $P(H) = p$, $P(H') = q$ and assume, for contradiction, that $p + q > 1$; i.e. $q > 1 - p$. Consider the lottery (p, x^*); assume that it is generated by the roulette wheel gamble (R, x^*) for $R = [0, p)$. We have $H \sim_l R$. Also $H' \succ_l$ *not*-R because $(H', x^*) \sim (q, x^*) \succ (1-p, x^*)$; the latter strict preference follows from $q > 1 - p$ and expected utility for lotteries.

Consider the gamble $(H \cup H, x^*) = ($*not*-$(H \cup H'), x_*; H', x^*; H, x^*)$, and compare it to the roulette wheel lottery $(\varnothing, x_*;$ *not*-$R, x^*; R, x^*)$ (which is the degenerate lottery yielding x^* for sure). Because $H' \succ_l$ *not*-R and $H \succeq_l R$, by event dominance the former gamble is strictly preferred to the latter. That means, however, that $(H' \cup H, x^*) \succ x^*$, contradicting the monotonicity Axiom 3.3. Stage 1 has been established.

Stage 2. For disjoint horse events H, H', $P(H \cup H') = P(H) + P(H')$.

Consider the act $(H \cup H', x^*)$. We rewrite the act as $($*not*-$(H \cup H'), x_*; H, x^*; H', x^*)$. Assume $P(H) = p$, $P(H') = q$. Compare the act just-described to the lottery $(1 - p - q, x_*; p, x^*; q, x^*)$. By Stage 1, the latter lottery can be defined indeed. Assume that the lottery is generated by roulette wheel events $R = [0, p)$, $R' = [p, p+q)$, and *not*-$(R \cup R') = [p+q, 1)$; i.e. the lottery corresponds to $($*not*-$(R \cup R'), x_*; R, x^*; R', x^*)$. Now $H \sim_l R$, $H' \sim_l R'$, and two-fold application of event dominance, once with $\preceq$ and once with $\succeq$, implies that the lottery is indifferent to the act $($*not*-$(H \cup H'), x_*; H, x^*; H', x^*)$. We can rewrite the lottery and act to obtain $(p+q, x^*) \sim (H \cup H', x^*)$. That implies that $P(H \cup H') = p + q$. Stage 2 has been established.

Stage 3. Acts are valued by their SEU value.

Consider an act $(H_0, x_0; \ldots; H_n, x_n)$. Assume that $P(H_j) = p_j$ for all j. As in the proof of Theorem 3.8, the p_j's sum to 1, hence we can consider the lottery $(p_0, x_0; \ldots; p_n, x_n)$, corresponding to some $(R_0, x_0; \ldots; R_n, x_n)$. We have $H_j \sim_l R_j$ for all $j \geqq 1$, therefore by two-fold application of event-dominance the gamble and the lottery are indifferent. This implies that the value of the gamble is its SEU value. ||

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