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An Explanation and Characterization for the Buying of Lotteries

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Abstract

Popular lotteries typically give a very small probability to win a large prize and a moderate chance to win smaller prizes. In this paper, a rank dependent model is axiomatized, with an S-shaped weighting function, capable of giving an account for the popularity of these lotteries. Also, the role of utility, loss aversion and scale compatibility in the explanation of the buying of lotteries is discussed.

Introduction

The history of mankind shows that people like to gamble. Gambling, however, has posed problems to economic theory. Expected utility theory does not provide the possibility for giving a plausible account, as will be explained below. A solution is to question if gambling behavior, although systematically observed, does at all fall under the realm of rational behavior. In this vein, gambling has often been explained in non-economic terms, by the enjoyment or the production of adrenaline that gambling can provide. However, also economic characteristics affect the attractiveness of gambles. A typically popular lottery, while actuarially unfair, has two distinctive features: first, it gives a very small chance to win a large amount and second, it gives a considerable chance to win a small amount or to break even. On the other hand, an actuarially fair fifty-fifty gamble is almost always rejected.

This paper provides an axiomatic model that is based on rank dependent utility theory developed by Quiggin (1982). The model implies an S-shaped weighting function that gives an account of the characteristics of popular lotteries. Besides economic characteristics encompassed by the axiomatization, we readily concede that psychological factors play an important role. These are discussed in the final section.

Explaining the Buying of Lotteries

Theoretically, gambling is somewhat puzzling. From a broad point of view, expected utility does quite a good job at describing decisions under risk, mainly by the assumption of diminishing marginal utility. Under this assumption, however, people are predicted to dislike long shot lotteries (gaining a large amount with a small probability), which is contrary to observed behavior (Shapiro and Venezia, 1992). The popularity of long shot lotteries becomes even more puzzling if we accept the intuitively compelling idea that a reference-point is relevant in the evaluation of gambles (Markowitz, 1952, Kahneman and Tversky, 1979): Empirically it is well established that most people exhibit loss-aversion, which implies that the utility is steeper for losses than for gains. But by gambling, people do not seem to attach as much importance to the ticket-fee as a steep utility for losses would suggest.

Friedman and Savage (1948) also tried to explain the attractiveness of lotteries within an expected utility framework. They hypothesize a utility with a convex region, to account for risk-seeking behavior[1]. When people have an initial wealth located near the first inflection point (with the convex region at the right hand side), people are predicted to reject fifty-fifty fair gambles but to accept long shot gambles. Still, this hypothesis is not very convincing. The predictions are only accurate for a specific range of initial wealths. If wealth is in the convex region, subjects are predicted to take fifty-fifty gambles, even if the gambles are actuarially unfair to a moderate degree. As Quiggin (1991) points out, the levels of wealth in the convex region are predicted not to be found in the society, for at these levels people will gamble until they reach a level of wealth in one of the concave regions. Although this argument may seem contrived, it points at a major weakness of the Friedman-Savage hypothesis. The attractiveness of typical lottery formats does not seem to be related to the level of wealth of participants as predicted by the Friedman-Savage hypothesis. For more elaborated criticisms, see Machina (1982).

A suggestion already given by Edwards (1962) is that not only the attitude towards money is important in decision making, but also the attitude towards probability. To explain gambling and the buying of insurance simultaneously, it is hypothesized that people are prone to overestimate the probability of rare events. For very small probabilities, the probability-distortion effect could outweigh the relative loss of utility. Also, by assuming underweighting of moderate and large probabilities, it is predicted that people dislike actuarially fair gambles with a moderate probability for obtaining the highest outcome.

Although transforming probabilities proves to be an adequate instrument to explain gambling and other kinds of systematic violations of expected utility, it is not easy to model. The first model, studied by Edwards and others, violates first order stochastic dominance, see for instance Fishburn (1978) or Wakker (1989). Quiggin (1982) was the first to find a proper model for using transformed probabilities in decision making, the rank dependent utility model. Quiggin (1991) used this model to explain the features of popular lotteries, through an S-shaped weighting function. Such a function had also been proposed by Karni and Safra (1990). A typical example is given in Figure 1.

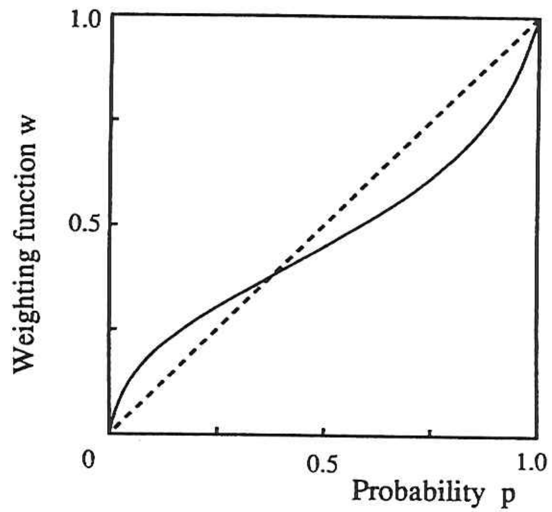

Figure 1. Example of S-shaped weighting function.

The next section provides an axiomatization of such weighting functions, that predict the overweighting of small probabilities and the underweighting of large probabilities. We concentrate there on the modeling of probability effects. Utility effects will not be axiomatized, but will be discussed in later sections.

A Characterization of the Buying of Lotteries

This section derives the characterization of S-shaped weighting functions when utility is linear, and is more technical than the other parts of the paper. We assume that the *outcome* set is an interval $[0, M]$, for a fixed $M > 0$. Thus, we only consider gain outcomes and sign-dependence is not an issue. By $\mathcal{P}$ we denote the set of all *gambles* over the outcomes, i.e., all probability distributions over $[0, M]$ that assign probability one to a finite subset of $[0, M]$. By $(x_1, p_1; \ldots; x_m, p_m)$ we denote the gamble that with probability p_1 results in outcome $x_1, \ldots,$ and with probability p_m in outcome x_m. For the study of rank-dependence, the topic of this paper, it is convenient to formulate a mechanism that can generate the randomness, i.e., to assume an underlying state space. We model it through the unit interval, as follows.

Assume that a number is picked at random from $[0,1]$. For each subinterval A of $[0,1]$, the probability $P(A)$ that the random number is contained in A is the length of A. This determines the usual uniform probability distribution over $[0,1]$. A gamble is generated by a random variable f on $[0,1]$, i.e., f denotes a mechanism that specifies for each number from $[0,1]$ an amount of money obtained when the number in question is the number randomly chosen from $[0,1]$.We use the term *act* instead of random variable, and the set of acts is denoted by $\mathcal{F}$; all acts are assumed measurable and take only a finite number of outcomes. Thus, each act generates a gamble, and for each gamble acts can be constructed to generate the gamble; we identify acts and gambles. Acts can be mixed, in a 'pointwise manner', as $\alpha f + (1-\alpha)g : \omega \mapsto \alpha f(\omega) + (1-\alpha)g(\omega)$. It is important to note that here *outcomes* are mixed, and not *probabilities*.

Finally, $\succeq$ denotes the *preference relation* of a decision maker on the gambles. We assume that each act is equivalent to the gamble it generates, thus $\succeq$ also denotes preferences over acts. By $\succ$ we denote strict preference, and $\sim$ denotes indifference. We assume throughout that $\succeq$ is *complete* ($f \succeq g$ or $g \succeq f$ for all acts f, g) and transitive, i.e., it is a *weak order*. We further assume that $\succeq$ satisfies *strict stochastic dominance*, i.e., if in $(x_1, p_1; \ldots; x_n, p_n)$ any of its outcomes that occurs with positive probability is increased, the resulting gamble is strictly preferred. Finally, we assume that $\succeq$ is continuous in both outcomes and probabilities. So a minor change in outcomes, as well as a minor change in probabilities, leads to a minor change in preference.

A function $V : \mathcal{F} \rightarrow \mathbb{R}$ *represents* $\succeq$ if, for all acts f, g,

$$f \succeq g \Longleftrightarrow V(f) \geq V(g).$$

We say that the *rank-dependent utility* model holds if there exist a strictly increasing continuous *weighting function* $w : [0,1] \rightarrow [0,1]$, with $w(0) = 0, w(1) = 1$, and a *utility function* $u : [0, M] \rightarrow \mathbb{R}$ such that $\succeq$ is represented by the following form, displayed now for the case $x_1 \geq \ldots \geq x_n$ and subsequently defined in general:

$$(x_1, p_1; \ldots; x_n, p_n) \mapsto \sum_{i=1}^{n} \pi_i u(x_i),$$

with π_i the difference $w(\sum_{j=1}^{i} p_j) - w(\sum_{j=1}^{i-1} p_j)$, which is $w(p_1)$ for $i = 1$. If the outcomes are not ordered as assumed above, then they are first permuted and then a formula as above is applied.

In rank-dependent utility, comonotonicity plays an important role; it was introduced in Schmeidler (1989). Acts f and g are *comonotonic* if there do not exist $\omega, \omega' \in [0,1]$ such that $f(\omega) > f(\omega')$ and $g(\omega) < g(\omega')$. In words, the acts do not order states in contradictory manners. A set of acts is *comonotonic* if every pair of acts in the set is comonotonic. Obviously, every constant act is comonotonic with every other act.

Rank-dependent utility was introduced in Quiggin (1982), under the special assumption that $w(1/2) = 1/2$. That assumption was subsequently criticized by economists. It was argued that "pessimism" would be the general phenomenon. Pessimism implies that a decision maker assigns relatively more importance in a decision to the relatively unfavorable outcomes of that decision. It can be modeled through convexity of the weighting function[2]. Indeed, it is readily seen that convexity of the weighting function means that differences $w(p + \epsilon) - w(p)$ are relatively smaller if p is smaller. Since our method of integration starts with the highest outcomes, this means that the highest outcomes receive relatively smaller decision weights $\pi_i = w(\sum_{j=1}^{i} p_j) - w(\sum_{j=1}^{i-1} p_j)$. Convexity of the weighting function cannot be satisfied under Quiggin's assumption that $w(1/2) = 1/2$, unless the trivial case of the identity-weighting function.

Psychological research has revealed, however, that pessimism and risk aversion are not universal phenomena. Rather, for small probability/large gain gambles, the majority of people tends to be risk seeking. This is exhibited for instance by the existence and popularity of gambles. It can be explained under rank-dependent utility by an S-shaped weighting function, that is concave on an interval $[0, p]$ and convex on an interval $[p, 1]$. Such a function can very well agree with Quiggin's (1982) assumption that $w(1/2) = 1/2$, although empirical research suggests that $w(1/2)$ is somewhat smaller than $1/2$.

In the ensuing formal analysis we restrict attention to linear utility functions. It is well understood that this assumption is not empirically realistic; still it is a useful working hypothesis to most, clearly bringing to the fore the characteristics of the weighting function. This is similar to Yaari's (1987) approach, where also linearity of utility was imposed.

Wakker (1990a) characterized rank-dependent utility, for uncertainty, with linear utility functions and either convex or concave weighting functions. This has also been done by Yaari (1987) and Chateauneuf (1991). The result of Wakker (1990a) was different because it characterized convexity/concavity directly in terms of a basic condition for rank-dependent utility: the comonotonicity condition. In view of the new insights in the empirically prevailing shape of the weighting functions, it seems warranted that the axiomatization of Wakker (1990a) be adapted to S-shaped weighting functions; that is the purpose of this section.

The following definition is similar to the independence condition for decisions under risk that underlies the utility result of von Neumann and Morgenstern (1944). There is, however, one essential difference, that is, in the condition below outcomes are mixed and not probabilities. We shall nevertheless use the same terminology as for probability mixtures, because the conditions can be identified in a mathematical sense; compare Wakker (1990a, Appendix). Also the condition is given in a comonotonic version.

Definition 1 *We say that $\succeq$ satisfies (mixture-)independence if, for all acts $\{f, g, h\}$ and $0 < \alpha < 1$,*

$$f \succ g \Rightarrow \alpha f + (1-\alpha)h \succ \alpha g + (1-\alpha)h. \qquad (1)$$

We say that $\succeq$ satisfies comonotonic independence if implication (1) is required only when $\{f, g, h\}$ are comonotonic.

Elucidation for comonotonic independence has been given in Section 4 in Wakker (1990a). It can readily be verified that mixture

independence, in the presence of the usual assumptions, characterizes expected value maximization, thus uniquely determines the preference relation. For instance this can be derived from the theorem below.

A version of the following theorem for uncertainty was given in Wakker (1990a, Theorem 5 and Appendix). Wakker (1990c) showed how to derive the present risk result from it. For uncertainty, very similar results are given in Schmeidler (1989), Wakker (1990b, Theorems 6, 11), and Chateauneuf (1991). For risk, similar results are provided in Yaari (1987), Weymark (1981, Theorem 3), Wakker (1987, Theorem 4.2), Chateauneuf (1990, 1991).

Theorem 1 *The following two statements are equivalent:*

(i) Rank-dependent utility holds, where the utility for money is linear.

(ii) The preference relation $\succeq$ satisfies comonotonic independence.

Further, the weighting function in (i) is uniquely determined.

It can readily be derived from Wakker (1990a, Corollary 8 and Lemma 10) that w in the above theorem is convex if and only if comonotonic independence is strengthened to the following condition: *pessimism-independence* holds if implication (1) is required only when g, h are comonotonic. Similarly, w is concave if and only if the following condition holds: implication (1) is required only when f, h are comonotonic (*optimism-independence*). Wakker (1990a) gave the results for uncertainty, but it is well known that the conditions obtained there are, for uncertainty, equivalent to convexity, or concavity respectively, of the weighting function w.

Next we turn to the condition that characterizes the S-shape for weighting functions. In this, $\mathcal{P}(M, p)$ is the subset of gambles that assign a probability of at least p to the maximal outcome M, and $\mathcal{P}(0, 1-p)$ denotes the subset of $\mathcal{P}$ of those gambles that assign a probability of at least $1-p$ to the zero outcome.

Lemma 1 *Suppose rank-dependent utility holds with linear utility. Then the weighting function is concave on $[0, p]$ if and only if p satisfies optimism-independence on the set $\mathcal{P}(0, 1-p)$, and it is convex on $[p, 1]$ if and only if p satisfies pessimism-independence on the set $\mathcal{P}(M, p)$.*

Proof. First consider the set $\mathcal{P}(0, 1-p)$, and let $(x_1, p_1, \ldots, x_n, p_n)$ be an element thereof. It can be transformed into another gamble, as follows: First, the probability for outcome 0 is decreased by $1-p$; then, this probability $1-p$ is distributed evenly over the remaining outcomes (which may still include 0), in other words, all remaining probabilities are multiplied by $1/p$. Through this transformation, the preference relation and the rank-dependent representation can be transferred into a new preference relation and rank-dependent representation on the entire ("isomorphic") set $\mathcal{P}$. The property of optimism-independence is carried over by this transformation; therefore, for the new rank-dependent representation, the weighting function w^* is concave on $[0,1]$ if and only if optimism-independence is satisfied on $\mathcal{P}(0, 1-p)$. Now $w(q) = w^*(q/p) \times w(p)$ for all $0 \leq q \leq p$, so concavity of w on $[0,p]$ holds if and only if optimism-independence holds on $\mathcal{P}(0, 1-p)$.

The result concerning the set $\mathcal{P}(M, p)$ can be derived similarly, now probability p is proportionally shifted from outcome M to the other outcomes, and one proceeds as above. □

We are now ready to formulate the condition that characterizes rank-dependent utility with linear utility and an S-shaped weighting function: $\succeq$ satisfies *S-shape independence* for probability p if implication (1) is required whenever either $\{f, g, h\}$ are comonotonic, or g, h are comonotonic and f, g, h assign probability p or higher to the outcome M, or f, h are comonotonic and f, g, h assign probability $1-p$ or higher to outcome 0.

Theorem 2 *The following two statements are equivalent:*

(i) Rank-dependent utility holds, where the utility for money is linear, and w is concave on $[0,p]$ and convex on $[p,1]$.

(ii) The preference relation $\succeq$ satisfies S-shape independence for probability p.

The weighting functions characterized above will be most regular and appealing if they are differentiable at the point p.

Discussion

The overweighting of small probabilities explains why people consider long shot gambles attractive. But popular lotteries are also typically characterized by various smaller prizes. The model with linear utility, characterized above, does not explain why such lotteries are preferred to single prize lotteries. If, however, we incorporate utility effects and a reference point effect, the presence of smaller prizes can be explained. To explain the idea, we compare a single prize lottery $(M, p; 0, 1-p)$ to a two prize lottery $(M, q; m, r; 0, 1-q-r)$. The probability r of winning a small prize is assumed to be around .15 for a typical lottery. If we assume a weighting function similar to Figure 1, this implies that the probability of winning the smaller prize is also overweighted[3]. Assume that both lotteries have the same expected value, so Mp equals $Mq+mr$. These lotteries can be expected to yield the same profit to the operator. According to the model with linear utility, the single prize lottery has a value of $w(p)M$ and the two prize lottery has value $w(q)M + [w(q+r) - w(q)]m$. The single prize lottery has the higher value, which is derived from concavity of w on the relevant interval.[4]

If we assume diminishing marginal utility for the outcomes of the lotteries, not yet characterized by the model (a characterization is provided Wakker and Tversky, 1993), the evaluation of the two lotteries becomes different. The two prize lottery becomes better, relative to the single prize lottery, because the smaller prize of the two prize lottery has a relatively higher utility than the top prize of the single prize lottery. Without specific assumptions about the weighting and utility function, however, no definite predictions can be made.

A second phenomenon not yet incorporated in the present model is loss aversion. Subjects tend to attach *much* more value to an amount of money that may be lost than to the same amount that may be gained. This results in a distinctive effect on the evaluation of both lotteries, to the favor of the two prize lottery. When participating in the single prize lottery, people are almost certain to lose their ticket fee. The two prize lottery, however, produces a reasonable chance to avoid a loss (while retaining the long shot effect). Because losses loom larger than gains, it is expected that people will prefer the reasonable chance to break even at least, to an almost certain loss. This we call the break even effect.

The break even effect is enhanced by the underweighting of moderate to large probabilities. This was pointed out by Quiggin (1991), and can be illustrated by treating the zero outcomes in the two exemplary lotteries as the outcomes denoting the loss of the ticket fee. The chance of losing the ticket fee in the two prize lottery is now $1-(q+r)$. This probability is underweighted if the chance to win a prize is overweighted, as is hypothesized, see Figure 1. For the one prize lottery the probability of losing is also underweighted by the same argument, but it will not result in a significant effect: the probability of losing still does not deviate much from unity. So probability distortion, modeled with an S-shaped weighting function, acts like a two-edged sword enhancing the popularity of a two prized, long shot lottery: on the one hand, the chance of winning the top prize or a smaller prize is overweighted, on the other hand the chance of losing the ticket fee is underweighted.

So far we have been arguing that a probability-distortion effect can explain why certain kinds of gambles are more popular than others, thus accommodating observed gambling behavior. But gambling behavior is complex, so it is to be expected that other psychological factors are relevant for explaining why people find lotteries attractive, even if these are actuarially far from fair. The already mentioned element of joy and excitement of gambling constitutes an important explanatory factor. We think that phenomena explaining the preference reversal effect also play a role in explaining the popularity of gambles, and turn now to a discussion of these.

The preference reversal effect, discovered by Slovic and Lichtenstein (1968), has been well established. Subjects are presented with a bet providing a high chance of getting a small amount of money (the P-bet) and a bet giving a small probability for a large amount of money (the \$-bet), mostly with a slightly higher expected value. When asked to choose between those two bets most people opt for the P-bet, but when they are asked to state their minimal selling prizes, they state a higher amount for the \$-bet. Moreover, Goldstein and Einhorn (1987) found that when subjects were asked to rank the two bets by the attractiveness of the bets, the P-bet was chosen far more often. But in terms of the minimum selling prize, the \$-bet is quite often ranked higher.

Tversky, Slovic and Kahneman (1990) show that the preference reversal phenomenon cannot be explained plausibly by violations of independence or reduction of compound lotteries. People rather violate procedure invariance: various seemingly equivalent elicitation-procedures for ranking gambles lead to different rank-orderings. One of the effects leading to violations of procedure invariance is scale-compatibility, see Tversky, Sattath and Slovic (1988). If people are asked to rank gambles according to their minimal selling prices, people will pay more attention to the value of the outcomes. This way to elicit preferences makes the \$-bet more attractive. If people are asked to make a choice, they make a more integrated evaluation, comparing the trade-off in probability and pay-off. This procedure thus leads to more attention for the probability distribution, enhancing the attractiveness of the P-bet.

People presumably do not order lotteries by a mechanism of pricing. For example, there does not exist a bargaining mechanism for lottery

tickets. The decision to buy a lottery ticket is a choice, where participation is preferred over abstaining. But typical lotteries exclusively direct attention to the outcomes: probabilities are never made explicitly available to the buyers. This prevents the buyer from making the integrated evaluation as done in the choice problems in preference reversal experiments. Instead, the buyer is led to evaluate the lottery on the only scale available, which causes her to overvalue the lottery. Research on the preference reversal effect shows that if people primarily pay attention to the money scale, they are prone to an overpricing of the \$-bet: Tversky, Slovic and Kahneman (1990) found that 83.9% of the subjects showing preference reversals overpriced the \$-bet.

We conclude that recent developments in decision theory improve our understanding of the buying of lotteries.

Notes

1. It should be noted that Friedman and Savage interchange the terms convex and concave compared to current terminology.
2. Here we emphasize that our way of integration is dual to the one most common in decisions under risk for rank-dependent utility; convexity of our weighting function is equivalent to concavity of the weighting function in the more common approach. We chose our way of integration because it is more common in decisions under uncertainty and has been used in cumulative prospect theory of Tversky and Kahneman (1992).
3. Formally, the smaller prize will be overweighted if $w(q+r) - w(q) > r$.
4. This can be shown as follows. We can rewrite the two prize lottery as $(M, bp; aM, (p-bp)/a; 0, 1-bp-(p-bp)/a)$, $0 < a, b < 1$, thus satisfying equality of expected value. The value of this lottery now becomes: $w(bp)M + aM[w((p-bp)/a + bp) - w(bp)]$. By concavity of w on the relevant interval, we conclude $w((p-bp)/a+bp) - w(bp) < w((p-bp)/a)$ and by concavity of w and $1/a > 1$ we find $w((p-bp)/a) < w(p-bp)/a$, so $w(bp)M + aM[w((p-bp)/a + bp) - w(bp)] < M[w(bp) + w(p(1-b)]$. By concavity of w on the relevant interval, $M[w(bp) + w(p(1-b))]$ is smaller than $Mw(p)$, which is the rank dependent value of the single prize lottery.

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