Independence of Irrelevant Alternatives and Revealed Group Preferences

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Choice functions are considered which assign a choice to every choice situation. A choice function reveals a binary relation on the set X of all possible choices. Under certain restrictions on the choice function this binary relation is representable by a function with appealing properties defined on the set of all choices.

In this paper X is the strictly positive orthant of n-dimensional Euclidean space, and choice situations are the nonempty convex compact subsets of X. One may think of a consumer having to choose from a possibly nonlinear budget set representing all combinations of n commodities that are available to this consumer, or of (a group of) n bargainers having to choose from a set of feasible compromises. The choice function then is the consumer's demand function, or the bargainers' solution: in both cases it is assumed to be a single-valued map which assigns to every choice situation a feasible choice.

Let Σ denote the collection of all choice situations, and let the choice function $\psi : \Sigma \to X$ be given (so $\psi(S) \in S$ for every $S \in \Sigma$). We say that $x \in X$ is *directly revealed preferred to* $y \in X$ if there is an $S \in \Sigma$ with $x = \psi(S)$ and $y \in S$; we then write xRy, suppressing dependence of the binary relation R on ψ. A binary relation Q on X *represents* the choice function ψ if for every choice situation S we have $\{\psi(S)\} = \{x \in S : xQy \text{ for every } y \in S\}$, i.e. ψ uniquely maximizes Q on S. The choice function ψ satisfies *Independence of Irrelevant Alternatives* (IIA) if for all choice situations S and T with $S \subset T$ and $\psi(T) \in S$ we have $\psi(T) = \psi(S)$. The IIA axiom was first proposed in Nash (1950), in the context of bargaining game theory. The following elementary result very clearly shows what IIA means:

GAME THEORY AND APPLICATIONS

THEOREM 1. *The choice function ψ can be represented by a binary relation Q if and only if ψ satisfies IIA.*

It follows that the directly revealed preference relation (of the consumer, or the 'group' of bargainers) represents ψ if and only if ψ satisfies IIA. Let us write xPy $(x, y \in X)$ if there is an $S \in \Sigma$ with $x = \psi(S)$ and $y \in S$, $y \neq x$. We call P the *directly revealed strict preference relation*. Further, we call ψ *Pareto-optimal* (PO) if for no $S \in \Sigma$ there is a $y \in S$ with $y \geq \psi(S)$, $y \neq \psi(S)$.

The following four theorems all hold for dimension two, i.e. $n = 2$. Below, we will devote a paragraph to the case $n > 2$.

THEOREM 2. *Let $n = 2$, let ψ satisfy PO and IIA, and suppose $aPbPc$ for some $a, b, c \in X$. Then cRa.*

In other words, for dimension two, if ψ satisfies PO and IIA, then the corresponding revealed preference relation does not contain *cycles* of length smaller than four. Unfortunately, IIA and PO are not sufficient to exclude cycles of length four, as can be shown by a (rather complicated) example. Thus, in order to exclude cycles of any length, i.e. to have ψ satisfy the well known *Strong Axiom of Revealed Preference*, we need some additional condition on ψ. We say that ψ satisfies *Pareto-continuity* (PC) if for every sequence $S, S_1, S_2, \ldots \in \Sigma$ with $S_k \to S$ and $P(S_k) \to P(S)$ (limits taken with respect to the Hausdorff metric) we have $\psi(S_k) \to \psi(S)$. We say that ψ is *continuous* if the condition $P(S_k) \to P(S)$ can be omitted. We then have the following result:

THEOREM 3. *Let $n = 2$, and let ψ satisfy PO, IIA, and PC. Then ψ satisfies SARP.*

So the addition of PC to the list of axioms in Theorem 2 suffices to exclude cycles of any length in the revealed preference relation, if the dimension is two; in that case, the binary relation Q of Theorem 1 can be taken transitive. Let f be a real valued function on X; we say that the choice function ψ maximizes f if $f(\psi(S)) > f(x)$ for every $S \in \Sigma$ and $x \in S$, $x \neq \psi(S)$. The three axioms in Theorem 3 do not imply the existence of a function maximized by ψ (e.g. let ψ assign to each choice situation the lexicographically maximal point); again we need to strengthen the conditions on ψ. This leads to the main result of the paper:

THEOREM 4. *Let $n = 2$, and let ψ be a Pareto-optimal continuous choice function. Then the following two statements are equivalent:*

(*a*) *ψ satisfies IIA.*
(*b*) *ψ maximizes a real valued function f on X.*

Let us call a real valued function f on X *strongly monotonic* if $f(x) > f(y)$ whenever $x \geq y$, $x \neq y$. Let us call f *strongly quasiconcave* if the set $\{y \in X : f(y) \geq f(x)\}$ is strictly convex for every $x \in X$, where we call $T \subset X$ strictly convex if $\lambda x + (1 - \lambda)y$ is an interior point of T whenever $x, y \in T$, $x \neq y$, $0 < \lambda < 1$. Our final theorem shows that a function f as in Theorem 4(b) cannot be too irregular (although it may be impossible to have f continuous).

THEOREM 5. *For a choice function ψ the following two statements are equivalent:*

(a) *ψ is continuous and satisfies PO and IIA.*
(b) *ψ maximizes a strongly monotonic strongly quasiconcave real valued function f on X.*

All the proofs in the paper are self-contained, except for the proof of the downwards implication in Theorem 4 which uses a result by Jaffray (1975). Of course, there are numerous relations with the (other) literature on revealed preference; a few references are given below.

If the dimension is greater than two ($n > 2$), then even PO, IIA, and continuity of the choice function do not imply SARP any more. This is shown by extending an example of Gale (1960) for the linear case to our general convex sets. Theorems 4 and 5 still hold if we replace IIA by SARP.

Under stronger conditions, related results for the case of separable f and with variable dimension were obtained by Lensberg (1987).

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