In: H.J.M. Peters & O.J. Vrieze (1987, Eds.), *Surveys of Game Theory and Related Topics*, 163-180, CWI Tract 39, Centre for Mathematics and Computer Science, Amsterdam.

CHAPTER VI

FROM DECISION MAKING UNDER UNCERTAINTY TO GAME THEORY

by Peter Wakker

1. INTRODUCTION

From a mathematical point of view many results from game theory and decision making under uncertainty are equivalent. An example is the characterization, as the class of "balanced" games, of the class of cooperative games with side-payments which have nonempty core. This was found by Shapely (1967); earlier Bondareva (1963, in Russian) had obtained this result; see also Driessen (1985, section 2.8). In Huber (1981, Lemma 10.2.2) the same result, obtained independently, is given for the context of decision making under uncertainty. Many other results have been formulated for one of the two contexts, but seem to be as interesting when formulated for the other context. One such example, not elaborated in this paper, is the theory of "belief functions" of Shafer (1976), formulated for the context of decision making under uncertainty. We think that notions such as the "degree of internal conflict" of a belief function, as developed by Shafer, are of utmost interest when studied for game theory. For a concise introduction into the basic concepts of Shafer's theory, see Zang (1986, section 1).

This paper presents new approaches to several topics in game theory. The obtained results have in common that they have been derived, by simple translation algorithms, from results on probability theory and decision making under uncertainty. Section 5 will show how this was done. Further in section 5 proofs will be indicated.

The aim of this paper is to show the usefulness of the adopted translation algorithms.

2. ORDERING COALITIONS IN COOPERATIVE GAME THEORY

First we present the basic definitions of the theory of cooperative games with side payments. Let $N = \{1,\ldots,n\}$ be a nonempty finite set of *players*, and 2^N the set of *coalitions*. A function $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0, S \supset T \Rightarrow v(S) \geqq v(T)$, and $v(N) = 1$ is called *characteristic function*; the second (monotonicity) condition, and the third (normalizing) condition, are not generally assumed in literature, but for convenience will be assumed throughout this paper. The quantity $v(S)$ may designate for instance the power, or earnings, or (negative) costs of a coalition S, or the number of publications of S in the International Journal of Game Theory; in this paper $v(S)$ will be called the *worth* of the coalition S. An element $x = (x_1,\ldots,x_n) \in \mathbb{R}^n_+$ is an *allocation*, and is interpreted as a function, assigning x_j to player j, for all j. In this paper $\mathbb{R}_+$ is the set containing 0 and all positive real numbers. The quantity x_j may for instance stand for money. If $\Sigma_{j=1}^n x_j = v(N)$, x may be interpreted as a division of the worth of the "grand" coalition $\{1,\ldots,n\}$, and x is called *efficient*. A central question in the theory of cooperative games with side payments is the question which efficient allocation is "fair" for a characteristic function v. The usual procedure to determine this is to compare the amount $x(S) := \Sigma_{j\in S} x_j$, allocated to the coalition S, with the worth $v(S)$ of the coalition S, and,for instance, to take as a criterion that every $x(S)$ should be at least as large as the worth of the coalition. In that case x is called a *core* allocation.

This paper will propose criterions of a different character. The idea of our criterions will be that the central notions to be considered are the *orderings* of coalitions as induced by the worths and allocations, and not the worths and allocations themselves. As an example where this may be natural think of the many cases, e.g. in politics, where an (inefficient) allocation (5,5,...,5) over a set of persons with equal worth is preferred over an allocation (6,7,6,7,...), simply because the second allocation would induce "unjust" inequalities, and tensions. As a second example think of definitions of wealth which say that a person is rich if she (or he) belongs to the 20 percent of most wealthy persons in her country. Again it is the ordering induced by allocated money which is relevant, not the absolute

amounts of money.

Let us now consider some criteria of the new kind. They all express the idea that more worthy coalitions should get allocated more.

DEFINITIONS 2.1. An allocation x is :

Almost agreeing with v if, for all coalitions S,T,
$[v(S) \geqq v(T) \Rightarrow x(S) \geqq x(T)]$;

Strictly agreeing with v if, for all coalitions S,T,
$[v(S) > v(T) \Rightarrow x(S) > x(T)]$;

Agreeing with v if, for all coalitions S,T,
$[v(S) \geqq v(T) \Leftrightarrow x(S) \geqq x(T)]$.

The first criterion above might be called "socialistic" since it allows for the occurrence of coalitions S,T with $x(S) = x(T)$ while $v(S) > v(T)$, whereas $v(S) = v(T)$ will always imply $x(S) = x(T)$; thus equality is increased by it. The second criterion might be called "capitalistic" since it allows for the occurrence of coalitions S,T with $x(S) > x(T)$ while $v(S) = v(T)$, whereas $v(S) > v(T)$ will always imply $x(S) > x(T)$. Obviously an allocation is agreeing if and only if it is both strictly agreeing and almost agreeing. Also one elementarily verifies that x is almost agreeing with v if and only if, for all coalitions S,T, $[x(S) > x(T) \Rightarrow v(S) > v(T)]$, and strictly agreeing with v if and only if for all coalitions S,T, $[x(S) \geqq x(T) \Rightarrow v(S) \geqq v(T)]$.

It will be observed that not for every characteristic function v there exist agreeing allocations x. For example let $N = \{1,2,3\}$, and let v assign 1/3 to every one-player coalition, 1/2 to $\{1,2\}$, and 2/3 to every other two-player coalition. Then, to be agreeing, x will have to assign the same to every one-player coalition, which will imply $x\{1,2\} = x\{2,3\}$; however $v\{1,2\} < v\{2,3\}$ should imply $x\{1,2\} < x\{2,3\}$. The characteristic function just described does not satisfy the following condition (set $S = \{1\}$, $T = \{3\}$, $V = \{2\}$ in the definition below):

DEFINITION 2.2. The characteristic function v is *ordinally additive* if, for all coalitions S,T,V with $S \cap V = \emptyset = T \cap V$:
$v(S) \geqq v(T) \Leftrightarrow v(S \cup V) \geqq v(T \cup V)$.

It is straightforwardly verified that this condition is necessary for the existence of an agreeing allocation. Still, it turns out not to be sufficient, as the following example shows.

EXAMPLE 2.3. (Kraft, Pratt & Seidenberg). Let $N = \{1,2,3,4,5\}$, and let $v(\emptyset) = 0$, $v\{1\} = 2/32$, $v\{2\} = 3/32$, $v\{3\} = 4/32$, $v\{1,2\} = 5/32$, $v\{1,3\} = 6/32$, $v\{4\} = 7/32$, $v\{1,4\} = 8/32$, $v\{2,3\} = 9/32$, $v\{5\} = 10/32$, $v\{1,2,3\} = 11/32$, $v\{2,4\} = 12/32$, $v\{3,4\} = 13/32$, $v\{1,5\} = 14/32$, $v\{1,2,4\} = 15/32$, $v\{2,5\} = 16/32$, $v\{1,3,4\} = 17/32$, $v\{3,5\} = 18/32$, $v\{2,3,4\} = 19/32$, $v\{1,2,5\} = 20/32$, $v\{1,3,5\} = 21/32$, $v\{4,5\} = 22/32$, $v\{1,2,3,4\} = 23/32$, $v\{1,4,5\} = 24/32$, $v\{2,3,5\} = 25/32$, $v\{1,2,3,5\} = 26/32$, $v\{2,4,5\} = 27/32$, $v\{3,4,5\} = 28/32$, $v\{1,2,4,5\} = 29/32$, $v\{1,3,4,5\} = 30/32$, $v\{2,3,4,5\} = 31/32$, $v(N) = 1$.

It is straightforwardly checked that this v is a characteristic function which satisfies ordinal additivity. Still, no agreeing allocation x exists since the inequalities $x\{1\} + x\{3\} < x\{4\}$, $x\{1\} + x\{4\} < x\{2\} + x\{3\}$, $x\{3\} + x\{4\} < x\{1\} + x\{5\}$, $x\{2\} + x\{5\} < x\{1\} + x\{3\} + x\{4\}$, when added up, reveal a contradiction.

In the above example there does exist an almost agreeing efficient allocation, viz. (1/16,2/16,3/16,4/16,6/16). There do exist characteristic functions for which no almost agreeing efficient allocation exists, and characteristic functions for which no strictly agreeing allocation exists, whereas these characteristic functions do satisfy ordinal additivity. Further it can be seen that for all cooperative games with side payments with less than five players, ordinal additivity is sufficient for the existence of agreeing allocations. For all cooperative games with side payments with less than six players ordinal additivity is sufficient for the existence of an almost agreeing efficient allocation. The reader may want to check these facts by writing a computer program on his personal computer which checks all cases.

The necessary and sufficient conditions for the existence of the several kinds of agreeing allocations can be obtained by standard applications of theorems of the alternative, (see for instance Scott, 1964), and are as

follows, with $x \leqq y$ if $x_j \leq y_j$ for all j, $x \gg y$ if $x_j > y_j$ for all j, and $x \underset{\neq}{>} y$ if $x_j \geq y_j$ for all j, and $x \neq y$.

THEOREMS 2.4. *There exists an almost agreeing efficient allocation if and only if : For every pair of sequences of coalitions* $(S_1,\dots,S_n)$ *and* $(T_1,\dots,T_n)$ *for which every player occurs in more coalitions in the left sequence than in the right*

$$not(v(S_1),\dots,v(S_n)) \leqq (v(T_1),\dots,v(T_n)). \tag{2.1}$$

There exists a strictly agreeing efficient allocation if and only if : For every pair of sequences of coalitions $(S_1,\dots,S_n)$ *and* $(T_1,\dots,T_n)$ *for which every player occurs in at least as many coalitions in the left sequence as in the right*

$$not(v(S_1),\dots,v(S_n)) \gg (v(T_1),\dots,v(T_n)). \tag{2.2}$$

There exists an agreeing efficient allocation if and only if : For every pair of sequences of coalitions $(S_1,\dots,S_n)$ *and* $(T_1,\dots,T_n)$ *for which every player occurs in the same number of coalitions in the left sequence as in the right*

$$not(v(S_1),\dots,v(S_n)) \underset{\neq}{>} (v(T_1),\dots,v(T_n)). \tag{2.3}$$

□

Obviously the third condition in the theorem has to imply ordinal additivity of v. Note that the only property of v, used in our analysis, has been the way v orders the coalitions. Thus we might also have taken an ordering of the coalitions, instead of v, as primitive in our analysis. Note that without the efficiency restriction there always exists an almost agreeing allocation : (0,...,0). For agreeing allocations, and strictly agreeing allocations, $v\{1,\dots,n\}$ is positive, so x can always be normalized, and the requirement of efficiency in the above theorem does not induce any restriction, so might have been omitted.

We end this section with a conjecture : there exists a characteristic function which is ordinally additive, which has both an almost agreeing and a strictly agreeing allocation but no agreeing one.

3. BANKRUPTCY PROBLEMS

Let $n \in \mathbb{N}$ be fixed, $n \geq 3$. Let $E \in \mathbb{N}$ be fixed, and let $d = (d_1,\dots,d_n) \in \mathbb{N}_0^n$. E is an amount of money, to be divided among n players (or claimants) $1,\dots,n$ where each player j has advanced a claim of d_j. For any d, by d_+ we denote the total amount Σd_j of claims. A *division rule* $f : \mathbb{N}_0^n \to \mathbb{R}^n$ is a function which assigns to every claim $d = (d_1,\dots,d_n)$ a sequence of proportions $(f_1(d),\dots,f_n(d))$, with $f_j(d) \geq 0$ for all j, and $\Sigma f_j(d) = 1$, such that player j will receive a portion $f_j(d) \times E$ of the amount E. Obviously one might think of other interpretations, e.g. where d_j reflects the salary of a person j, and $f_j(d)$ the tax which the person is to pay; also d_j may stand for investment, one-player-coalition-worth, etc. Our set-up differs from the usual set-ups such as Aumann, R.J. & M.Maschler (1985), Moulin (1985a,b), Curiel, I., M. Maschler & S.H. Tijs (1986), and Young (1987) in considering only natural numbers as claims (and amounts) to be divided, and in leaving out of the analysis variability of the amount E to be divided.

We shall now proceed to consider some conditions for division rules.

DEFINITION 3.1. We call f *monotone* if, for all i and d :

$$f_i(d_1,\dots,d_i+1,\dots,d_n) > f_i(d_1,\dots,d_i,\dots,d_n).$$

So if a player can increase her (or his) claim, it will give her a larger portion of the amount E.

DEFINITION 3.2. We call a player i *uninvolved* for the division rule f if, for all $j \neq i \neq k$, and all d :

$$f_i(d_1,\dots,d_j+1,\dots,d_k,\dots,d_n) = f_i(d_1,\dots,d_j,\dots,d_k+1,\dots,d_n).$$

So if a player i is uninvolved, she has no interest in a replacement of part of the claim of player j to another player k. Her proportion $f_i(d)$ will depend only on her own claim d_i and the total claim $(d_+ - d_i)$ of the other players, as is easily verified. It protects player i against a manipulation of the remaining players to increase the sum of their shares by redistributing amongst each other the sum of their claims. Moulin (1985a)

introduced the condition that all players shall be uninvolved in a related context and called it "No Advantageous Reallocation".

DEFINITION 3.3. We call a pair of players i,j *proportionally uninvolved* if, for all $i \neq k \neq j$:

$$\frac{f_i(d_1,\dots,d_k+1,\dots,d_n)}{f_j(d_1,\dots,d_k+1,\dots,d_n)} = \frac{f_i(d_1,\dots,d_k,\dots,d_n)}{f_j(d_1,\dots,d_k,\dots,d_n)}$$

where one denominator being zero is to imply that the other denominator is zero too; in the presence of monotonicity that can only happen if $d_j = 0$

So then the proportion of the portions that player i and j receive from E depends only on d_i and d_j, and is independent of the other claims. This condition is somewhat stronger (in also restricting f_i/f_j if $f_i + f_j$ varies) than the consistency property as introduced in Kolm (1976, in a context with varying number of players and nonrational claims and amounts E); see also the consistency condition in Moulin (1985b). Now we characterize the division rules with the above properties.

THEOREM 3.4. *For a division rule* f *the following two statements are equivalent :*

(i) There exist nonnegative constants $\gamma_1,\dots,\gamma_n$, summing to one, and a nonnegative constant λ, such that for all i :

$$f_i : d \mapsto \frac{(\lambda\gamma_i + d_i)}{(\lambda + d_+)}.$$

(ii) The division rule f satisfies monotonicity, every player is uninvolved, and every pair of players is proportionally uninvolved. □

Note that f_i as in (i) above can be considered to be a convex combination of the amount γ_i that player i would receive if no player would have claimed anything, and d_i/d_+, the share of the total amount of claims that has been advanced by player i, with weights respectively λ and d_+. An indication of a full proof is provided in subsection 5.2. Let us just sketch here a way of proof. It is straightforward that statement (i) above implies (ii). So we assume (ii), and derive (i). First one determines the constants $\gamma_1,\ldots,\gamma_n$ as $f_1(0,\ldots,0),\ldots,f_n(0,\ldots,0)$. Next one calculates λ from $f_1(1,0,\ldots,0) = (\lambda\gamma_1 + 1)/(\lambda + 1)$. Note that the division rule as defined in (i) above is one division rule with the mentioned values $f_1(0,\ldots,0),\ldots,f_n(0,\ldots,0)$ and $f_1(1,0,\ldots,0)$, satisfying all conditions of (ii). Finally the most involved part of the proof is to demonstrate that the above-mentioned values of f, together with the mentioned conditions, uniquely determine all values f(d) with $d_+ = 1$, next those with $d_+ = 2$; by induction with respect to d_+, the uniqueness of f(d) follows for all d. In this the monotonicity condition serves to prevent that certain equalities will reduce to the trivial 0 = 0.

4. BETTER AND WORSE ALLOCATIONS

As in the previous sections, we consider in this section the question of how to choose between several possible allocations $(x_1,\ldots,x_n)$ over n players. And, as in section 2, a characteristic function v will occur in our analysis. Still the approach of this section will be different, and in Theorem 4.2 the status of observability of v will differ from the usual set-up in the theory of cooperative games with side payments.

Let us first sketch the approach by means of "Choquet integrals" , central for this section. For simplicity of exposition we shall assume that an arbitrator will finally decide which of a set of available allocations to choose.

4.1. *The choquet-integral-approach*

The approach of this subsection will be split up in six stages.

Stage 1. The arbitrator concentrates for a moment on one available allocation x.

Stage 2. For this allocation x, the arbitrator takes a permutation π on 1,...,n such that : $x_{\pi(1)} \geq x_{\pi(2)} \geq \ldots \geq x_{\pi(n)}$. So $\pi(1)$ is the richest player under allocation x, $\pi(2)$ the one-after-the-richest-player, etc. Note that we have not specified the way in which equally-rich players are to be ordered according to π. They may be ordered in any arbitrary way,

the approach sketched in the sequel will be such that this ordering is immaterial.

Stage 3. The players will enter, one by one, a room where the arbitrator is. First the richest player $\pi(1)$ enters, then $\pi(2)$, etc.

Step 3.1. After entrance of player $\pi(1)$ the arbitrator pays to $\pi(1)$ the amount $x_{\pi(1)} - x_{\pi(2)}$ that $\pi(1)$ receives more than player $\pi(2)$.

Step 3.2. Next player $\pi(2)$ enters the room, and the arbitrator pays to $\{\pi(1),\pi(2)\}$ the amount $x_{\pi(2)} - x_{\pi(3)}$ that $\pi(1)$ and $\pi(2)$ still are to receive more than player $\pi(3)$.

...

Step 3.i. Next player $\pi(i)$ enters the room, and the arbitrator pays to the present players $\pi(1),\ldots,\pi(i)$ the amount $x_{\pi(i)} - x_{\pi(i+1)}$ that the present players still are to receive more than player $\pi(i+1)$.

...

Step 3.n. Finally player $\pi(n)$ enters, all players are present now, and get payed the remaining amount $x_{\pi(n)}$.

Stage 4. Now that the payment in stage 3 has been fixed for every step, at every step the payment is valued by its product with the worth of the involved group of players.

Stage 5. The allocation x is valued by adding up all valuations of Stage 4, to give, with $x_{\pi(n+1)} := 0$:

$$\Sigma_{i=1}^{n}[(x_{\pi(i)} - x_{\pi(i+1)}) \times v\{\pi(1),\ldots,\pi(i)\}] \tag{4.1}$$

If we consider an allocation x as a function, assigning x_j to every player j, then the value in (4.1) is the *Choquet integral* of x with respect to the characteristic function v, see for instance Wakker (1986a, formula VI.2.5). Indeed, if v happens to be "additive", then (4.1) reduces to the usual integral.

Stage 6. For all available allocations a valuation is determined as it was for x above. Then the allocation with maximal valuation is chosen. If there are more allocations where the maximal valuation is attained, from these an arbitrary choice is made. If the supremum value of the valuations is not attained by any allocation, then some allocation is chosen which is close enough to the supremum in some sense. If the set of available allocations is compact, then the maximum will always be attained for some allocation.

4.2. *A characterization by ordering allocations*

In this section we characterize the approach of subsection 4.1. The method of characterization will differ from that of section 2. In this section we assume that the arbitrator takes a binary ("preference") relation $\geqq$ on $\mathbb{R}^n_+$, the set of all allocations. Here $x \geqq y$ means that the arbitrator would be willing to choose x if only x and y were available, i.e. she (or he) considers x at least as good as y. Next we consider conditions which will characterize the approach of subsection 4.1.

The binary relation $\geqq$ is a *weak order* if it is *complete* (i.e. for all x,y in $\mathbb{R}^n_+$: $x \geqq y$ or $y \geqq x$) and *transitive* (i.e. for all x,y,z in $\mathbb{R}^n_+$: if $x \geqq y$ and $y \geqq z$ then $x \geqq z$). As usual we write $x > y$ if $x \geqq y$ and not $y \geqq x$, $x \approx y$ if $x \geqq y$ and $y \geqq x$, $x \leqq y$ if $y \geqq x$, and $x < y$ if $y > x$. Further $\geqq$ is *strictly monotonic* if, for all allocations x,y,$[x > y \Rightarrow x > y]$, and $\geqq$ is *continuous* if, for all allocations y, the sets $\{x \in \mathbb{R}^n_+ : x \geqq y\}$ and $\{x \in \mathbb{R}^n_+ : x \leqq y\}$ are closed.

We call a pair of allocations x,y *comonotonic* if for no players i,j simultaneously $x_i > x_j$ and $y_j > y_i$. This is exactly the case where in Stage 2 of subsection 4.1 there can be chosen a same permutation π for x and y, see Wakker (1986a, Lemma VI.3.5, (i) $\Leftrightarrow$ (iii)). A set of allocations is *comonotonic* if every pair of allocations in the set is comonotonic. The main characterizing condition will be :

DEFINITION 4.1. The binary relation $\geqq$ satisfies *comonotonic independence* if for all comonotonic x,y,z and $\alpha \in (0,1)$ we have :

$$[x > y \Rightarrow \alpha x + (1-\alpha) z > \alpha y + (1-\alpha) z].$$

With this we get :

THEOREM 4.2. *For the binary relation $\geqq$ on the set of allocations the following two statements are equivalent :*

(i) There exists a characteristic function v such that, for all allocations x,y, $x \geqq y$ if and only if the Choquet integral of x is at least as large as that of y.

(ii) The binary relation $\geqq$ is a continuous strictly monotonic weak order which satisfies comonotonic independence.

Furthermore the characteristic function v is uniquely determined. □

So, if the approach of subsection 4.1 applies, then $\geqq$ satisfies the conditions mentioned in statement (ii) above, and reversedly, if $\geqq$ satisfies the conditions in statement (ii) above, then *there exists* a characteristic function v such that by means of this the approach of subsection 4.1 applies. The implication (ii) $\Rightarrow$ (i) above is mainly interesting in contexts where the characteristic function v is not easily available. As an example think of the case where players are ministers in a government, who during some years have been choosing among allocations of money over their departments. From their choices we can reveal "group preferences" of the form $x \geqq y$; if these preferences satisfy the conditions in statement (ii) above, then according to the above theorem we can derive from the choices of the ministers the characteristic function v. Then for any group S of ministers v(S) can be interpreted as an index for the power of this group of ministers.

5. THE RECIPE FOR THE ABOVE RESULTS, AND LITERATURE

The results presented in the previous sections were simple translations of results, formulated before in literature for decision making under uncertainty. The following translation has been involved everywhere :

Replace *state of nature* by *player*. (5.1)

5.1. *The translation algorithm of section 2*

In section 2 we translated results from a field of decision making under uncertainty which goes under the heading of "comparative probability theory". In comparative probability theory one considers a "more probable than" relation $\geq$ on subsets (events) of the state space $\{1,\ldots,n\}$, and one searches for a probability measure agreeing in some way with the more-probable-than relation. So, besides the already mentioned translations, the following translations are involved :

Replace *event* by *coalition* (5.2)

Replace *S is more probable than T* by *the worth of S is higher than that of T* (5.3)

Replace *probability* by *allocation*. (5.4)

The condition of ordinal additivity has been introduced by de Finetti (1931). For a long time it was not known whether this condition would suffice, in presence of some "natural" presumptions, to guarantee the existence of an agreeing allocation/probability measure, see for instance Savage (1954, page 40/41). The matter was settled by Kraft, Pratt & Seidenberg (1959), who provided the Example 2.3, and gave the necessary and sufficient condition of (2.3). Their work used an algebraic notation which may not be easily accessible for every reader. Later Scott (1964) sketched a general procedure to use theorems of the alternative or separating hyperplane theorems to solve inequalities such as those involved in section 2, for finite state/player spaces. Since then, the conditions as in Theorem 2.4 are well-known. Jaffray (1974a,b) gave a more general approach by which also inequalities for infinite state/player spaces can be solved; by means of this technique Chateauneuf (1985) obtained necessary and sufficient conditions for the existence of an agreeing probability measure/allocation for general state/player spaces. The author of this paper studied the topic in Wakker (1979) and Wakker (1981), mainly for infinite state/player spaces. In Wakker (1981, Theorem 4) it was indicated that, with ordinal additivity presupposed, the characterization of almost agreement in Theorem 2.4 also holds for infinite state/player spaces. Also Gilboa (1985) considered questions of this nature; in his work a nonadditive characteristic function was interpreted as a distortion of an additive probability measure. A recent and complete overview of

comparative probability theory is provided in Fishburn (1986).

5.2. *The translation algorithm of section 3*

The results of section 3 were obtained by translating work of Carnap on inductive reasoning, see Carnap (1962), Carnap & Jeffrey (1971), Fine (1973, section VII.D), Stegmüller (1973), Koerts & De Leede (1973), or Zabell (1981). As an example let us suppose that a die has been thrown several times. In this subsection $1,\ldots,n$ are the sides of the die; after every throw exactly one side ("state of nature") will come up. Further : $d = (d_1,\ldots,d_n)$ describes the number of times that the several sides have been observed after d_+ throws. And $f_j(d)$ designates the conditional subjective probability (Carnap preferred the interpretation as logical probability) that the $(d_+ + 1)$-th throw will give side j up, given the result of the previous throws. So the following translations are involved :

Replace *claim of player j* by *number of previously observed occurrences of side j of the die* (5.5)

Replace *proportion for player j* by *conditional probability for side j of the die* (5.6)

Like us, Carnap assumed monotonicity; so a new observation of a side of the die makes a next occurrence of this side more probable. And like us, Carnap assumed uninvolvedness of every side/player. Instead of our proportional uninvolvedness Carnap assumed "exchangeability", i.e. the probability of a sequence of outcomes depends only on the number of occurrences of the several sides of the die, and is independent of the particular order in which these sides occurred. This is equivalent to the equality, for all d,i,j :

$$f_i(d_1,\ldots,d_i,\ldots,d_j,\ldots,d_n) \times f_j(d_1,\ldots,d_i+1,\ldots,d_j,\ldots,d_n) =$$
$$f_j(d_1,\ldots,d_i,\ldots,d_j,\ldots,d_n) \times f_i(d_1,\ldots,d_i,\ldots,d_j+1,\ldots,d_n),$$

since the left-side gives the conditional probability that, given d, first side i will come up, next side j, whereas the right-side deals with the reversed order of occurrence of i and j.

In the presence of the other conditions, exchangeability is equivalent to proportional uninvolvedness of every pair i,j. Let us only show the derivation of exchangeability from proportional uninvolvedness, plus uninvolvedness. For any k such that $i \neq k \neq j$ (such a k exists since $n \geqq 3$)

$$\frac{f_i(d_1,..,d_i,..,d_j,..,d_n)}{f_i(d_1,..,d_i,..,d_j+1,..,d_n)} = \frac{f_k(d_1,..,d_i,..,d_j,..,d_n)}{f_k(d_1,..,d_i,..,d_j+1,..,d_n)} =$$

$$\frac{f_k(d_1,..,d_i,..,d_j,..,d_n)}{f_k(d_1,..,d_i+1,..,d_j,..,d_n)} = \frac{f_j(d_1,..,d_i,..,d_j,..,d_n)}{f_j(d_1,..,d_i+1,..,d_j,..,d_n)}$$

where the first equality follows from proportional uninvolvedness of i,k, the second from uninvolvedness of k, and the third from proportional uninvolvedness of j and k. The equality of the first and fourth quotient imply the equality given above as an equivalent of exchangeability.

Carnap showed that his conditions are equivalent to statement (i) in Theorem 3.4 (see for instance Zabell, 1981). This, together with the just derived observations, gives an alternative proof for our Theorem 3.4. The author studied Carnap's work for its applicability in probability calculations for the protection of statistical data files against anonimity disclosure, see Wakker (1986b).

5.3. *The translation algorithm of section 3*

The work of section 3 was obtained by translating work of Schmeidler for decision making under uncertainty, see Schmeidler (1984a,b,c). As an example, suppose a horse race will be held. There will participate n numbered horses, j is the "state of nature" that horse j will win. An *act* $x \in \mathbb{R}^n_+$ is a function from the states of nature to $\mathbb{R}_+$, interpreted as an investment (or bet, or whatever) that will result in a net gain of x_j if horse j will win. Now $\geqq$ denotes the preference relation of a decision maker over the set of acts, $x \geqq y$ meaning that the decision maker considers x to be at least as good as y. The characteristic function v is now interpreted as a nonadditive subjective probability measure for the decision maker; the higher v(S), where now S is an event, the more probable S is considered to be by the decision maker. So now the following translations are involved :

Replace *event* by *coalition* (5.7)

Replace *arbitrator* by *decision maker* (5.8)

Replace *allocation* by *act* (5.9)

Replace *characteristic function* by *subjective nonadditive probability* (5.10)

Schmeidler (1984a) showed the equivalence of statements (i) and (ii) in Theorem 4.2 in a slightly different context; in his work payment was not in money, as in section 4 above, but in lotteries over some set. The generalization of Schmeidler's work to the case where payment is in terms of elements of a "mixture space" (see for instance Wakker, 1986 a, Definition VII.2) is completely straightforward. One example of mixture spaces is the case of sets of lotteries over another set, as in Schmeidler's work; another example is $\mathbb{R}_+$, as in section 4 above. Hence in a mathematical sense Theorem 4.2 is completely analogous to Schmeidler (1984a, The Theorem) The author made use of Schmeidler's work on nonadditive probabilities in Wakker (1986a, Chapter VI).

6. CONCLUSION

This paper is based on the observation that the same mathematical structure is underlying many problems in decision making under uncertainty and in game theory. By simple translations, mainly by interchanging "state of nature" and "player", many results derived for decision making under uncertainty and game theory can be interchanged. This paper gave some examples. Admittedly, sometimes, such as in Definition 3.3, a minimal amount of creativity was needed. Still, an author in lack of inspiration, but in need of publications, may succeed with the following algorithm :

Take any theorems from a journal dealing with the topic of game theory, or probability theory/decision making under uncertainty.

Carry out the translations as described in this paper.

Send the resulting theorems to a journal dealing with the other topic than the original journal.

Do not refer to the original journal.

Do not refer to this paper.

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