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EXTENDING MONOTONE AND NON-EXPANSIVE MAPPINGS BY OPTIMIZATION

by

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SUMMARY

We demonstrate a theoretical application of optimization theory. We use it to prove theorems on the extendeability of the domain of non-expansive, and (strictly) monotone, mappings (under preservation of the characteristic property), by formulating the key problem in it as an optimization problem.

1. INTRODUCTION

In 1934 it was shown for the first time that a non-expansive mapping G from a subset of $\mathbb{R}^m$ to $\mathbb{R}^m$, can be extended to a non-expansive mapping G' from the whole of $\mathbb{R}^m$ to $\mathbb{R}^m$, by Kirszbraun [10]. In 1943 and 1945 Valentine [21, 22], unaware of this, rediscovered the result, and extended it to (infinite-dimensional) Hilbert spaces. The key problem in this all is to extend a finite domain of a non-expansive mapping on $\mathbb{R}^m$ by just one point. The rest then follows from the compactness of spheres w.r.t. the weak topology, the finite intersection property for compact sets, and the Lemma of Zorn. The most efficient proof for the key result (solving the key problem) has been given by Schoenberg [19] in 1953.

The same matters can be studied for monotone mappings. Of course, also for strictly monotone mappings, and also for mappings, satisfying a Lipschitz condition; but these all follow from the non-expansive case. For monotone mappings the key-problem, extending a finite domain by just one point, has first been solved in 1962 by Minty [13]. Much more cannot be done here, in general the domain of a monotone mapping does not have to be extendable.

In 1962 Grünbaum gave a tricky and short proof of a result that entails the key results mentioned above both for the non-expansive and the monotone case [7]. Eversince many generalizations have been obtained. In this paper however we mainly restrict our attention to the key problem.

Non-expansive and monotone mappings find applications in the solvation of equations, such as nonlinear integral equations, differential equations, and they play a role in the theory of variational inequalities (for a survey and references see [2, 16, 17]).

The key problems concern the solvation of inequalities. In the next sections we show how to formulate them as optimization problems, to be able to derive them from the basic theorems of optimization theory. To this end all but one of the inequalities are handled by an induction hypothesis, and then the remaining one is handled by optimization.

2. THE KEY PROBLEM FOR NON-EXPANSIVE MAPPINGS

By $x'y$ we denote the usual inner product on $\mathbb{R}^m$, and by $\|x\| = \sqrt{x'x}$ the usual Euclidean norm. For $V \subset \mathbb{R}^m$ we denote by conv(V) the convex hull of V, by int(V) the interior of V, and by rint (V) the relative interior of V (i.e. the interior of V w.r.t. the affine hull of V).

For the key problem we consider a mapping G from a finite n-point subset $\{a^0, \ldots, a^{n-1}\}$ of $\mathbb{R}^m$, to $\mathbb{R}^m$, where we write b^j for $G(a^j)$ for all j, and we assume G is *non-expansive*, i.e. $\|G(x) - G(y)\| \leq \|x-y\|$ for all x,y in the domain of G. For simplicity one may assume $a^i \neq a^j$ if $i \neq j$. The issue of this section is whether the domain of G can be extended to some arbitrary point a, i.e. whether there exists b such that $\|b-b^j\| \leq \|a-a^j\|$ for all j. We shall make use of induction to show that such a b always exists. To show this for n = 1 is simple, take $b = b^0$. Next we make, for some $n \in \mathbb{N}$, the induction hypothesis, that for every n-point set, and every non-expansive G on it, the domain of G can be extended by one arbitrary point. And we now have to show for some non-expansive G with n+1-point domain $\{a^0, \ldots, a^n\}$, where $b^j := G(a^j)$ for all $0 \leq j \leq n$; and for some arbitrary a, that there exists b such that $\|b-b^j\| \leq \|a-a^j\|$ for all $0 \leq j \leq n$. To this end we define, for every $1 \leq j \leq n$, $f_j : \mathbb{R}^m \to \mathbb{R}$ by $f_j(y) = \|y-b^j\|^2 - \|a-a^j\|^2$, and we define $F : \mathbb{R}^m \to \mathbb{R}$ by $F(y) = \|y-b^0\|^2$, for all y. Our task is to find a b such that $f_j(b) \leq 0$ for all $1 \leq j \leq n$, and $F(b) \leq \|a-a^0\|^2$. Therefore we consider the optimization problem :

Find $y \in \mathbb{R}^m$ with : $f_j(y) \leq 0$ for all $1 \leq j \leq n$; F(y) min !

The induction hypothesis guarantees that the *set of feasible points* V, i.e. $V := \{y \in \mathbb{R}^m \mid f_j(y) \leq 0 \text{ for all } 1 \leq j \leq n\}$, is non-empty. Also it is closed and bounded, so the continuous F has a minimum on it. For b we thus take a feasible point where F is minimal. (Since the f_j's are (strictly) convex, and F is strictly convex, this b is unique. It is the projection of b^0 on V. We shall not make use of this.) Remains to be shown that $F(b) \leq \|a-a^0\|^2$.

Let $J := \{j \mid 1 \leq j \leq n, f_j(b) = 0\}$, the indices of the "active restricting" f_j's at b. Let, for $f : \mathbb{R}^m \to \mathbb{R}$ differentiable at y, $\nabla f(y)$ denote the gradient of f at y. Now we can apply a basic result of optimization theory, the so-called "Fritz John necessary conditions" must be satisfied in b, i.e. there must exist non-negative real numbers $(p_j)_{j \in J \cup \{0\}}$, summing to 1, such that $p_0 \nabla F(b) + \Sigma_{j \in J} p_j \nabla f_j(b) = 0$, see Theorem 5.1.3 of [1]. This means $p_0(b-b^0) + \Sigma_{j \in J} p_j(b-b^j) = 0$. Once this is established, the supposition $\|b^0-b\| > \|a^0-a\|$ will give a contradiction. To see this we assume a = b = 0, which can be obtained by translating $\{a^0, \ldots, a^n, a\}$ and $\{b^0, \ldots, b^n, b\}$, or by substituting $a^{j'} = a^j - a$ for a^j, and 0 = a-a for a; and $b^{j'} = b^j - b$ for b^j, and 0 = b-b for b; for all j; and by then leaving out dashes. Let us resume what we have now.

We know $\|b^j\| = \|a^j\|$ for all $j \in J$, and (suppose) $\|b^0\| \overset{(1)}{>} \|a^0\|$ (so $b^0 \neq 0$). Also $\|b^i-b^j\| \leq \|a^i-a^j\|$ for all i,j. Application of the formula $x'y = 1/2(\|x\|^2 + \|y\|^2 - \|x-y\|^2)$

reveals $b^{i'}b^j \geq a^{i'}a^j$ for all $i, j \in J$; even $b^{0'}b^j \overset{(2)}{>} a^{0'}a^j$ for all $j \in J$. From this we also see $\|\Sigma_{j \in J} p_j a^j\|^2 = \Sigma_{i,j \in J} p_i p_j a^{i'} a^j \overset{(3)}{\leq} \|\Sigma_{j \in J} p_j b^j\|^2$. Furthermore we know $p_0 + \Sigma_{j \in J} p_j = 1$, $p_0 b^0 = -\Sigma_{j \in J} p_j b^j$. Now combining this all we get $p_0\|b^0\|^2 = (p_0 b^0)'b^0 = (-\Sigma_{j \in J} p_j b^j)'b^0 \leq$ (by(2)) $(-\Sigma_{j \in J} p_j a^j)'a^0 \leq \|\Sigma_{j \in J} p_j a^j\| \, \|a^0\| \leq$ (by (3)) $\|\Sigma_{j \in J} p_j b^j\| \, \|a^0\| \leq$ (by (1)) $\|\Sigma_{j \in J} p_j b^j\| \, \|b^0\| = p_0\|b^0\|^2$. Since $\|b^0\| > 0$, we know that, if $p_0 > 0$, then $\|p_0 b^0\| = \|\Sigma_{j \in J} p_j b^j\| > 0$ so the last inequality is strict, otherwise $p_0 = 0$ so at least one p_i is positive, and the first inequality is strict. Thus we always have $p_0\|b^0\|^2 > p_0\|b^0\|^2$, i.e. a contradiction. We have demonstrated :

Theorem 2.1. (Key result for non-expansive mappings.)

If G is a non-expansive mapping from a finite subset of $\mathbb{R}^m$ to $\mathbb{R}^m$, then its domain can be extended by any arbitrary point under preservation of non-expansiveness.

3. THE KEY RESULT FOR MONOTONE MAPPINGS

In this section we consider a mapping M from a subset of $\mathbb{R}^m$ to $\mathbb{R}^m$, that is *monotone*, i.e. $[M(x) - M(y)]'(x-y) \geq 0$ for all x and y in the domain of M. Note that for $\mathbb{R}^1$ this means M is non-decreasing, thus monotone. And we assume here that the domain of M is a finite set, such as an n-point set $\{a^0, \ldots, a^{n-1}\}$. We write b^j for $M(a^j)$, for all j. Then the key problem is, whether the domain of M can be extended to some arbitrary a, under preservation of monotonicity of course. I.e., can a b be found such that $(a-a^j)'(b-b^j) \geq 0$ for all $0 \leq j \leq n-1$? The key result says it can.

For n = 1 this is obvious, take $b = b^0$. Next make the induction hypothesis that the key result is proved for some $n \in \mathbb{N}$, every n-point domain can be extended. Then we prove the same for n+1. Thus let M be a monotone mapping with n+1-point domain $\{a^0, \ldots, a^n\}$, where $b^j := M(a^j)$ for all j, and let $a \in \mathbb{R}^n$. Does there exist b such that $(a-a^j)'(b-b^j) \geq 0$ for all j ? If $a = a^j$ for some j one takes $b = b^j$. So we assume $a \neq a^j$ for all j. Also we assume a = 0. This may be done since translation of $\{a, a^0, \ldots, a^n\}$ does not affect the problem. We may replace a, and every a^j, by 0 = a-a, and a^j-a.

Let A be the mxn matrix with columns $a^1, \ldots, a^n$, and let v be the element of $\mathbb{R}^n$ with j-th coordinate $b^{j'}a^j$ for all $1 \leq j \leq n$. Since a = 0, we must find b such that $b'A \leq v'$ and $b'a^0 \leq b^{0'}a^0$. (We consider vectors as columns, and denote the transpose by a dash. We write $x \leq y$ if $x_j \leq y_j$ for all j.) To this end we consider the optimization problem :

Find $y \in \mathbb{R}^m$ such that : $y'A \leq v'$, $-a^{0'}y$ max !

By our induction hypothesis the set of feasible outcomes $\{y \in \mathbb{R}^m \mid y'A \leq v'\}$ is non-empty. If the above optimization problem is unbounded, i.e. the maximum is ∞, then certainly a feasible y can be found such that $y'a^0 \leq b^{0'}a^0$, and we can let b be such an y. So assume the problem is bounded. Then the *solution set* $W := \{y \in \mathbb{R}^m \mid y \text{ is a feasible outcome where } -a^{0'}y \text{ is maximal}\}$ is non-empty. Let b be an element of this set. Our hope is that $-a^{0'}b \geq -a^{0'}b^0$.

The dual of the above, "prime", problem is :

Find $x \in \mathbb{R}^n$ such that : $Ax = -a^0$, $x \geqslant 0$, $v'x$ min !

For this dual problem we apply the fact that $(a^i - a^j)' (b^i - b^j) \geqslant 0$, i.e. $a^{i'}b^i + a^{j'}b^j \geqslant a^{i'}b^j + a^{j'}b^i$ for all $0 \leqslant i,j \leqslant n$, we define $x_0 = 1$, and get for any feasible x :

$$(\Sigma_{i=0}^n x_i)(\Sigma_{j=0}^n x_j a^{j'} b^j) = \Sigma_{i=0}^n x_i^2 (a^{i'}b^i) + \Sigma_{i<j} x_i x_j (a^{i'}b^i + a^{j'}b^j) \geqslant \Sigma_{i=0}^n x_i^2 (a^{i'}b^i) +$$

$$\Sigma_{i<j} x_i x_j (a^{i'}b^j + a^{j'}b^i) = \Sigma_{i=0}^n \Sigma_{j=0}^n x_i x_j (a^{i'}b^j) =$$

$$(\Sigma_{i=0}^n x_i a^i)' (\Sigma_{j=0}^n x_j b^j) = (a^0 + Ax)' (\Sigma_{j=0}^n x_j b^j) = 0.$$

We conclude $\Sigma_{j=0}^n x_j (a^{j'}b^j) \geqslant 0$, i.e. $v'x \geqslant -a^{0'}b^0$. Thus the value of the dual problem is $\geqslant -a^{0'}b^0$. By the known duality theorems (see for instance Chapter 17 of [5]; many other texts require that rank $A = m$, and $n > m$; this however is not essential for the duality Theorems), the value of the dual problem equals that of the prime. Thus $-a^{0'}b \geqslant -a^{0'}b^0$, and we have reached our purpose : $a^{0'}b \leqslant a^{0'}b^0$. Thus the key result is proved. It is also a known thing that a non-empty solution set W for the prime problem is bounded if and only if $(a=)\ 0 \in U :=$ int conv $\{a^0, \ldots, a^n\}$. [If for $\varepsilon > 0$ $B_{2\varepsilon}(0)$, the closed sphere with centre 0 and radius 2ε, is contained in U, then, for $y \in W$, $\|y\|$ is bounded since $(\varepsilon y/\|y\|)' y \leqslant \max \{a^{0'}b^0, a^{1'}b^1, \ldots, a^{n'}b^n\}$; in the other case 0 is on the boundary of, or outside of, U; and $y + \lambda w \in W$ for every $y \in W$, $\lambda > 0$ and existing $w \neq 0$ with $w'a^j \leqslant 0$ for all j.] Thus we get :

Theorem 3.1.

If M is a monotone mapping from a finite subset of $\mathbb{R}^m$ to $\mathbb{R}^m$, then its domain can be extended to one arbitrary point under preservation of monotonicity. The set of possible image points for the new point in the domain is bounded if and only if the new point is in the interior of the convex hull of the original domain.

Proof

The boundedness of the set W' of possible image points is characterized in a way identical to the boundedness of the solution set W above, just replace W by W' in the reasoning between brackets [...] above. Boundedness, for points in the interior of the convex hull, can also be obtained by Theorem 1 in [18]. The rest has been done above.

4. RESULTS DERIVED FROM THE KEY RESULT FOR NON-EXPANSIVE MAPPINGS

The following theorem is a strengtening of Theorem 2.1.

Theorem 4.1.

If G is a non-expansive mapping from a subset S of a Hilbert space E, to a Hilbert space F (e.g. $E = F = \mathbb{R}^m$), then there exists a non-expansive extension G' of G, $G' : E \to F$. G' is continuous.

Proof

First we show that the domain S can be extended to include one arbitrary point $a \notin S$. For this a we must find b such that $\|b - G(s)\| \leqslant \|a - s\|$ for all $s \in S$, i.e.

$b \in V := \cap_{s \in S} B_{\|a-s\|}(G(s))$, where $B_\varepsilon(x)$ denotes the closed sphere with centre x and radius ε, for $\varepsilon > 0$. If such a b would not exist the intersection V would have to be empty. But since all involved spheres are compact w.r.t. the weak topology, by the finite intersection property there then must be a *finite* number of elements $s^1, \ldots, s^n$ of S, such that $\cap_{j=1}^n B_{\|a-s^j\|}(G(s^j)) = \emptyset$. The points $a, s^1, \ldots, s^n$ are contained in an $n+1$-dimensional subspace, isomorphic to $\mathbb{R}^{n+1}$ (if necessary by adding "artificial" dimensions). Also $G(s^1), \ldots, G(s^n)$, in F, are contained in a finite-dimensional subspace F' of F; by adding artificial dimensions this F' can be thought to be contained in an $n+1$-dimensional Hilbert space, isomorphic to $\mathbb{R}^{n+1}$. By Theorem 2.1 in this space a point b can be found such that $\|b - G(s^j)\| \leqslant \|a - s^j\|$ for all j. But then the projection b' of b on F' must be in $\cap_{j=1}^n B_{\|a-s^j\|}(G(s^j))$, contradicting the emptyness of this set. We conclude that the domain of G can always be extended by one point. That it can also be extended to the whole of E, is now a straightforward consequence of the Lemma of Zorn. If E is separable this extension can also be done constructively, by extending the domain first, point after point, to a denumerable dense subset of E, and then by continuity to the whole of E. Continuity of G of course is straightforward. □

The above result has first been obtained in [22]. In more general contexts it does not have to be valid. As an example think of a not-complete inner product space E, and let F be its completion. Let $(b^j)_{j=1}^\infty$ be a sequence of elements of E, converging to $b \in F$, such that $b \notin E$. And let $G : \{b_j \in F \mid 1 \leqslant j < \infty\} \to E$ be identity. Then G cannot be extended to the whole of F, because it cannot be extended to b.

Also in Banach spaces that are not (isomorphic to) Hilbert spaces things can go wrong. For instance let $E = F = \mathbb{R}^4$, with norm $\|(x_1, \ldots, x_4)\| = \sup\{|x_1|, |x_2|\} + \sqrt{x_3^2 + x_4^2}$, so E and F are the product space of $\mathbb{R}^2$ with supnorm, and $\mathbb{R}^2$ with Euclidean norm. Let $G(-1,0,0,0) = (0,0,-1,0)$, $G(0,\sqrt{2},0,0) = (0,0,0,1)$, $G(1,0,0,0) = (0,0,1,0)$. Then no image point can be found for $(0,1,0,0)$. (Without $(0,\sqrt{2},0,0)$ in the domain, for $G(0,1,0,0)$ only $(0,0,0,0)$ could have been chosen.)

But Theorem 4.1 can, for instance, be proved for $\mathbb{R}^m$ with the supnorm. For more results see [12,6,20,15,4].

Definition

A mapping F satisfies a *Lipschitz condition* if there is $\gamma > 0$ such that $\|F(x) - F(y)\| \leqslant \gamma \|x - y\|$ for all x,y in the domain of F.

Definition

A mapping F from a subset A of a Hilbert space E, to E, is *strictly monotone* if $(x-y)' (F(x) - F(y)) > 0$ whenever $x \neq y$.

The following theorems are straightforward applications of Theorem 4.1.

Theorem 4.2.

If a mapping F from a subset of a Hilbert space, to another Hilbert space, satisfies a Lipschitz condition, its domain can be extended to the whole Hilbert space, preserving the same Lipschitz condition.

Proof

Let $\gamma > 0$ be such that $\|F(x) - F(y)\| \leqslant \gamma\|x-y\|$ for all x,y in the domain of F. Define $G := F/\gamma$, then G is non-expansive. Thus G can be extended to a non-expansive G' with the whole Hilbert space as its domain. Then $F' := \gamma G'$ yields the desired extension of F. □

Theorem 4.3.

Let F be a monotone mapping from a subset A of a Hilbert space E to E, and let $\gamma > 0$ exist such that $(F(x) - F(y))'\ (x-y) \geqslant \gamma\|F(x) - F(y)\|^2$ for all $x,y \in A$. Then there exists a monotone continuous extension F' of F, with domain E, such that $(F'(x) - F'(y))'\ (x-y) \geqslant \gamma\|F'(x) - F'(y)\|^2$ for all $x,y \in E$.

Proof

Define $G := 2\gamma F\text{-Id}$ (Id is identity mapping). Then G is non-expansive. Extend G to a non-expansive $G' : E \to E$, and then define $F' := (G' + \text{Id})/2\gamma$. Since G' is continuous, F' is too. □

Theorem 4.4.

Let F be a strictly monotone mapping from a finite subset A of a Hilbert space E to E. Then there exists a monotone continuous extension F' of F, with domain E.

Proof

Define $\gamma := \min_{x \neq y \in A} [(x-y)'\ (F(x) - F(y)]/\|F(x) - F(y)\|^2$, which is positive since A is finite. Then apply Theorem 4.3. □

The above theorem, for -F, has found application in [8]. Of course, in the above, substitution of -F for F gives analogous results. Also the results can be extended to complex Hilbert spaces.

A further subject of investigation may be mappings satisfying the inequality $(x-y)'\ (F(x) - F(y)) \geqslant \gamma\|F(x) - F(y)\|^2$ for $\gamma < 0$. (For $\gamma = 0$ we are in the monotone case, see sections 3 and 5.) For $\gamma < 0$ new situations occur, corresponding, by means of the transformations in the proof of Theorem 4.3 , to mappings G satisfying $\|x-y\| \leqslant \|G(x) - G(y)\|$; corresponding, by taking inverse function, to the problem to extend injective non-expansive mappings to bijective ones. Of course here the key problem is simple and non-informative; extending a finite domain by one point is achieved by simply taking the new image point sufficiently far away from the origin.

5. *RESULTS DERIVED FROM THE KEY PROBLEM FOR MONOTONE MAPPINGS*

A result such as Theorem 4.1 cannot be derived for monotone mappings. For instance let, in $\mathbb{R}^1$, $A = \{1-1/j \mid 1 \leqslant j < \infty\}$, and $M(1-1/j) = j$ for all j. Then the domain of M cannot be extended to 1, M(1) would have to be larger than any natural number. The key result is valid in any Hilbert space, because any finite domain of a monotone mapping, and its image, is contained in a finite-dimensional subspace, within which the key problem can be solve. Furthermore we have the following result.

Theorem 5.1.

Let M be a monotone mapping from a subset A of $\mathbb{R}^m$, to $\mathbb{R}^m$. Then M can be extended to a monotone M' with domain A ∪ [rint conv(A)].

Proof

First we consider int conv(A). Let $x \in$ int conv(A). We want to extend the domain of M to x, i.e. find y in $\bigcap_{a \in A} V(a)$, where for $a \in A$ we define $V(a) := \{z \in \mathbb{R}^m \mid (a-x)'\ (M(a)-z) \geqslant 0\}$. By Theorem 3.1 every finite number of V(a)'s has non-empty intersection. The non-emptyness of $\bigcap_{a \in A} V(a)$, and thus the existence of an y as desired, is guaranteed if we can find a finite subset $\{a^1, \ldots, a^n\}$ of A such that $\bigcap_{j=1}^n V(a^j)$ is bounded. (Then we can intersect every V(a) with this closed and bounded, thus compact, set, and apply the finite intersection property for compact sets.) To this end we take $a^1, \ldots, a^n \in A$ such that $a \in \text{int conv}\ \{a^1, \ldots, a^n\}$. For instance these $a^1, \ldots, a^n$ can be obtained in two steps. In the first step one takes a small polytope in int conv(A), with a in its interior. Then, according to the Theorem of Carathéodory, each vertex of this polytope is convex sum of m+1 elements of A. One then takes, in the second step, as $\{a^1, \ldots, a^n\}$ the union of all these m+1 elements of A. Then Theorem 3.1 guarantees boundedness of $\bigcap_{j=1}^n V(a^j)$.

Thus we can extend the domain of M to any arbitrary $x \in$ int conv(A). It is a straightforward consequence of the Lemma of Zorn to extend M thus to int conv(A). If int conv(A) is empty, but rint conv(A) is not, then conv(A) is contained in a "minimal" affine subspace. We then project every M(a) on this affine subspace, and use the above reasoning to extend M within this subspace. □

The above Theorem can also be obtained as a corollary of Theorem 2 at page 253 of [14].

Theorem 5.2.

If M is a monotone mapping from a finite subset of $\mathbb{R}^m$, to $\mathbb{R}^m$, it can be extended to a monotone $M' : \mathbb{R}^m \to \mathbb{R}^m$.

Proof

First we extend the domain of M, point by point, by means of Theorem 3.1 , (and the denumerable version of the Lemma of Zorn) to a denumerable dense subset Q of $\mathbb{R}^m$, then by Theorem 5.1 to the whole of $\mathbb{R}^m$. Note that by the separating hyperplane Theorem (int) conv(Q) must equal $\mathbb{R}^m$.

The above result can be formulated for any Hilbert space E, instead of $\mathbb{R}^m$. One then first extends the domain of M to the finite-dimensional linear subspace containing the finite domain and image of M, by the above Theorem. Next M is extended to the whole E, an arbitrary element of E gets assigned the same image as its projection on the finite-dimensionnal linear subspace.

An extension as in the above Theorem certainly does not have to be continuous, contrary to that in Theorem 4.4.

As an example we consider $\mathbb{R}^2$. F has a four point domain, M(0,0) = (0,0); M(0,1) = (-1,0); M(1,0) = (0,1); M(1/2,0) = (0,2). If we concentrate on the first three domain points, we see M rotates these three points around the origin by 90 degrees to the "left". Indeed it can be demonstrated that any monotone extension of M must rotate the whole interior of the triangle of these three points by 90 degrees to the left, around the origin. Thus to (1/2, 1/j) must be assigned (-1/j, 1/2) for every $3 \leqslant j \in N$. If we now consider the fourth origin point (1/2, 0), we see the extension never can be continuous.

In the literature also more general settings are considered. For instance usually more general bilinear forms are considered than the inner products, such as the natural pseudo-inner product when the image space and original space for M are a conjugate pair of reflexive Banach spaces. See [14, 3, 16, 17]. Since the "key problem" of this paper only has to deal with finite-dimensional problems, our results can be translated to the latter context. Also more general subsets of the Cartesian product ExF of two spaces, than just (graphs of) functions, are considered. Our results on the extension of domains are special versions of the problem of adding pairs of points to such subsets, thus showing they are not "maximal". See [13, 3, 18, 11].

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