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CONTINUOUS EXPECTED UTILITY FOR ARBITRARY STATE SPACES

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ABSTRACT

We characterize subjective expected utility maximization with continuous utility, extending Wakker (1984 a,b) to arbitrary, e.g. infinite, state spaces. In Savage (1954) the main restriction, P6, requires structure for the state space. Our main restriction, requiring continuity of the utility function, may be more natural in economic contexts, since it is based on topological structure of the consequence space, structure that usually is present in economic contexts anyhow. Replacing the state interpretation by a time interpretation yields a characterization for dynamic contexts.

1. INTRODUCTION

A really satisfactory characterization, with appealing conditions that are both necessary and sufficient for Subjective Expected Utility (SEU) maximization in the context of Decision Making Under Uncertainty (DMUU; with "unknown" probabilities) is not yet available in literature. SEU provides however the most used approach in DMUU. Hence derivations (giving only sufficient conditions) are useful. Derivations and characterizations are necessary in justifying (verifying) or criticizing (falsifying) the use of SEU-models, and in showing their limitations.

The most known derivation of SEU maximization is provided by Savage (1954). His main restrictive assumption is P6, which requires structure for the state space, e.g. this must be infinite. For the consequence space there are hardly restrictions, mainly will the utility function have to be bounded, see §14.1 in Fishburn (1970).

In economic contexts the consequence space is usually endowed with structure, e.g. it is $\mathbb{R}^m_+$, so topical structure then is present. Thus here derivations of SEU maximization, employing this structure, may have value. Such derivations are Wakker (1984a), requiring consequences to be real numbers, and Wakker (1984b), giving the extension to the case where the consequence space is any connected topological space, e.g. (a convex subset of) $\mathbb{R}^n$. Their main restrictive assumption requires the utility function on the consequence space to be continuous. They have a further restriction: the state space must be finite. This last restriction is removed in this paper. We do share with Savage (1954) some boundedness condition, in that we do not handle all unbounded alternatives (="acts"). Our utility function may be unbounded though.

Let us emphasize that in our set-up probability measures are *finitely* additive, by which we mean that they are not necessarily countably additive. The latter is considered a special continuity assumption, see section 6. For surveys of SEU-maximization, see Fishburn (1981) or Schoenmaker (1982).

We shall not use lotteries in our set-up. With lotteries, a wide range of results is available in Fishburn (1982).

2. ELEMENTARY NOTATIONS AND DEFINITIONS

Let I be a set, called *state space*, with generic element i, a *(possible) state (of nature)*. Exactly one element of I is the *true state*, the others are *untrue*. A decision maker T is uncertain about which of the states is true.

Let C be a connected topological space, with generic elements $\alpha,\beta,\ldots$. For topological definitions, see Kelley (1955). A reader, not interested in topological details, may simply take in mind that C is (a convex subset of) $\mathbb{R}^m$. This satisfies all topological requirements, made in this paper.

Elements of C^I, the set of all functions from I to C, are called *alternatives*. We prefer this term to the, for DMUU more usual, term "acts". An alternative x has i-th coordinate x_i, i.e. assigns x_i to state i; x can be imagined to yield consequence x_i if i is the, to T unknown, true state. For $\alpha \in C$, $\bar{\alpha}$ is the constant alternative, assigning α to every i.

To stay close to probability theory we generalize our set-up by introducing measure-theoretic structure. We assume that a field A on I, with generic elements A,B, called *events*, and a field $\mathcal{D}$ on C, with generic elements C, D, are given. Here $\mathcal{D}$ is assumed to contain all open and closed, and all one-point subsets of C. A reader, not interested in measure-theoretic details, may simply take in mind that A contains all subsets of I. Then all measure-theoretic requirements, made in the sequel, are satisfied, and can be ignored. E.g. the set F, introduced in the sequel, then simply is C^I. This also shows that the introduction of general measure-theoretic structure really is a generalization of the set-up. We shall in the sequel restrict our attention to (subsets of) $F \subset C^I$, where F is the set of all alternatives x that are ($\mathcal{D}$-A-)measurable, i.e. $\{i|x(i) \in D\} \in A$ for all $D \in \mathcal{D}$.

We consider a binary relation $\succcurlyeq$ on F, the preference relation of T. We write $x \succcurlyeq y$ for $(x,y) \in \succcurlyeq$, meaning that T thinks x "at least as good" as y. We write $x \preccurlyeq y$ for $y \succcurlyeq x$, $x \prec y$ or $y \succ x$ if $y \succcurlyeq x$ and not $x \succcurlyeq y$, $x \approx y$ if both $x \succcurlyeq y$ and $y \succcurlyeq x$. We say $\succcurlyeq$ is a *weak order* on a subset F' of F if it is complete and transitive on F' (meaning $x \succcurlyeq y$ or $y \succcurlyeq x$; and $x \succcurlyeq z$ if both $x \succcurlyeq y$ and $y \succcurlyeq z$; for all $x,y,z \in F'$). Then $\approx$ is an equivalence relation on F'.

A *partition* P of I is a finite array $(A_j)_{j=1}^n$ of *events*, mutually disjoint, with union I. We then write $\Sigma_{j=1}^n \alpha_j 1_{A_j}$ for the alternative, assigning α_j to every $i \in A_j$. This is just a suggestive notation, and does not designate an addition operation or skalar multiplication, since C does not have to be endowed with such operations. Any alternative is *simple* if it has a finite range. A simple $x \in F$ can always be written as above. F^s is $\{x \in F \mid x \text{ is simple}\}$. By F^b we denote $\{x \in F \mid \mu,\nu \in C \text{ exist s.t. } \overline{x_i} \succcurlyeq \bar{\mu} \text{ and } \bar{\nu} \succcurlyeq \overline{x_i} \text{ for all } i\}$. Its elements are called *strongly bounded*. If $\succcurlyeq$ is a weak order then $F^s \subset F^b$. Note , if I is $\mathbb{N}$, $C =]0,1]$, $x_i = 1/i$, then x is bounded in the usual sense, but not strongly bounded.

NOTATION. For any $\alpha \in C$, $x \in C^I$, and $A \subset I$, $x_{-A}\alpha$ is the alternative that assigns x_i to every $i \notin A$, and α to every $i \in A$.

Note that $x_{-A}\alpha \in F(F^s;F^b)$, whenever $A \in A$, $x \in F(F^s;F^b$ and $\succcurlyeq$ a weak order). Usually we write i for $\{i\}$, thus $x_{-i}\alpha = x_{-\{i\}}\alpha$.

DEFINITION 2.1 An event A is *inessential* if $x \approx y$ whenever $x_i = y_i$ for all $i \in A^c$, and $x,y \in F^s$. Otherwise A is *essential*.

3. FURTHER DEFINITIONS

The following property is an extension of the CCI property, introduced in Wakker (1984a), and is the main new tool in our analysis.

DEFINITION 3.1. Event A is *Cardinally Coordinate Independently (CCI) related to* event B(w.r.t. $\succcurlyeq$) if $v_{-A}\gamma \succcurlyeq w_{-A}\delta$ whenever $x_{-B}\alpha \preccurlyeq y_{-B}\beta$, $x_{-B}\gamma \succcurlyeq y_{-B}\delta$, and $v_{-A}\alpha \succcurlyeq w_{-A}\beta$, for all simple x,y,v,w.

The idea, in short, is that from $x_{-B}\alpha \preccurlyeq y_{-B}\beta$, $x_{-B}\gamma \succcurlyeq y_{-B}\delta$, we should conclude that the "strength of preference" (not defined formally, further elaborated in section 3 of Wakker (1984a)) of γ over δ is at least as large as that of α over β, for as far as B is concerned. For A to be CCI related to B, A must follow B in this. Thus, if $v_{-A}\alpha \succcurlyeq w_{-A}\beta$, then certainly *not* should $v_{-A}\gamma \prec w_{-A}\delta$, for weak orders $\succcurlyeq$.

The CCI relatedness relation on A certainly does not have to be symmetric, not even if $\succcurlyeq$ satisfies nice properties, as being a weak order. For weak orders $\succcurlyeq$, every inessential A is CCI related to every B $\in$ A. Under "usual" circumstances (such as continuity) CCI relatedness is transitive, and A is CCI related to an inessential B iff A is inessential.

DEFINITION 3.2. We say $\succcurlyeq$ is *simply-continuous* if, for any partition $(A_j)_{j=1}^n$ and any alternative $x = \Sigma_{j=1}^n \beta_j 1_{A_j}$, we have closedness of $\{(\alpha_1,\ldots,\alpha_n) \in C^n \mid \Sigma_{j=1}^n \alpha_j 1_{A_j} \succcurlyeq x\}$ and $\{(\alpha_1,\ldots,\alpha_n) \in C^n \mid \Sigma_{j=1}^n \alpha_j 1_{A_j} \preccurlyeq x\}$ w.r.t. the product topology on C^n.

One may also formulate the above as: the binary relation $\gtrsim$ on C^n, defined by $(\alpha_1,\ldots,\alpha_n) \gtrsim (\beta_1,\ldots,\beta_n)$ if $\Sigma_{j=1}^n \alpha_j 1_{A_j} \succcurlyeq \Sigma_{j=1}^n \beta_j 1_{A_j}$, must be closed w.r.t. the product topology. This "finite-dimensional" continuity assumption is not unusually strong since a finite-dimensional product topology is not coarser than other usual topologies. If C is an Euclidean space then the product topology is equivalent to the usual Euclidean topology, or the sup-norm topology. If C is a metric space, then the product topology is equivalent to the sup-metric topology, which is used for instance in Koopmans (1972).

The main topological complications occur for infinite dimensions.

Then the product topology is coarser than other usual topologies. Continuity w.r.t. this then is too strong an assumption for our purposes. It would imply countable additivity of the probability measure P to be derived yet, and boundedness of the utility function U to be derived. In section 5 we shall have to deal with infinite-dimensional aspects, for that we use:

DEFINITION 3.2. We say $\succcurlyeq$ is *constant-continuous* on $F' \subset C^I$ if $\{\alpha \in C \mid \bar{\alpha} \succcurlyeq x\}$ and $\{\alpha \in C \mid \bar{\alpha} \preccurlyeq x\}$ are closed for all $x \in F'$.

Again this assumption is fairly weak, and implied by the sup-metric continuity assumption of Koopmans (1972), also by continuity w.r.t. the producttopology on C^I. In fact the only consequence of it, that we shall use, is that there exists γ s.t. $\bar{\gamma} \approx x$, whenever $\bar{\alpha} \succcurlyeq x \succcurlyeq \bar{\beta}$ for some $\alpha,\beta \in C$.

4. RESULTS FOR SIMPLE ALTERNATIVES

LEMMA 4.1. Let there not exist two disjoint essential events. Let C be topologically separable and connected. Then are equivalent:

(i) There exists a finitely additive probability measure P on A, and a continuous $U : C \to \mathbb{R}$, s.t. $\Sigma_{j=1}^s \alpha_j 1_{A_j} \succcurlyeq \Sigma_{k=1}^t \beta_k 1_{B_k} \Leftrightarrow \Sigma_{j=1}^s P(A_j)U(\alpha_j) \geq \Sigma_{k=1}^t P(B_k)U(\beta_k)$ for all $\alpha_1,\ldots,B_t$.

(ii) $\succcurlyeq$ is a simply-continuous weak order on F^s.

Furthermore, if $x \approx y$ for all $x,y \in F^s$, then U must be constant, and P is arbitrary; and if $x \succ y$ for some $x,y \in F^s$, then P(A) = 1 for all essential A, P(A) = 0 for all inessential A, and U may be replaced by another continuous $\tilde{U}$ iff $\tilde{U} = \phi \circ U$ for a strictly increasing ϕ; whenever (i) applies. Finally, every event is CCI related to every essential event if (i) applies.

PROOF. (i) $\Rightarrow$ (ii) is straightforward. So we assume (ii), and derive (i). There exists no essential A iff $x \approx y$ for all $x,y \in F^s$. In this case we must let U be constant, and P any arbitrary finitely additive probability measure. So suppose now (ii) is valid, and there exists an essential event. Then I itself is essential. For any A, A^c at least one must be essential, otherwise $x \approx y$ would follow by comparance to z, equal to x on A, to y on

A^c, for all $x,y \in F^S$. We have assumed not both A and A^c are essential. Thus exactly one of them is essential. If A and B are inessential, then so is $A \cup B$, because for all $x,y \in F^S$ with $x_i = y_i \ \forall i \in A^c \cap B^c$, $x \approx y$ follows from comparance to z, equal to x on A and $A^c \cap B^c$, and equal to y on A^c. Thus the collection of all essential events is an "ultrafilter", (i.e. A,B essential $\Rightarrow A \cap B$ essential, A essential $\Leftrightarrow A^c$ inessential, I is essential). Thus we define $P(A) = 1$ for all essential A, $P(A) = 0$ for all inessential A. This yields a finitely additive probability measure P.

Now we define U. We define $\succcurlyeq'$ on C by $\alpha \succcurlyeq' \beta$ if $\bar{\alpha} \succcurlyeq \bar{\beta}$. Then $\succcurlyeq'$ is continuous, and by section 4.6 of Debreu (1959) we get a continuous $U : C \to \mathbb{R}$ such that $\alpha \succcurlyeq' \beta \Leftrightarrow U(\alpha) \geq U(\beta)$. This U can be replaced by any continuous $\tilde{U}$ of the form $\phi \circ U$, ϕ strictly increasing.

Now (i) is proved if we demonstrate that

$$x = \Sigma_{j=1}^{s} \alpha_j 1_{A_j} \succcurlyeq \Sigma_{k=1}^{t} \beta_k 1_{B_k} = y \Leftrightarrow \Sigma_{j=1}^{s} P(A_j)U(\alpha_j) \geq \Sigma_{k=1}^{t} P(B_k)U(\beta_k).$$

Of $A_1,\ldots,A_s$ exactly one is essential, say A_1. Thus $x \approx \overline{\alpha_1}$. Say analogously $y \approx \overline{\beta_1}$. We have: $x \succcurlyeq y \Leftrightarrow \overline{\alpha_1} \succcurlyeq \overline{\beta_1} \Leftrightarrow U(\alpha_1) \geq U(\beta_1) \Leftrightarrow \Sigma_{j=1}^{s} P(A_j)U(\alpha_j) \geq \Sigma_{k=1}^{t} P(B_k)U(\beta_k)$.

The remark "Furthermore...." follows from the above exposition, the remark "Finally...." follows by substituting (i) in Def.3.1.

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LEMMA 4.2. Let $\succcurlyeq$ be a simply-continuous weak order. Let every event A be CCI related to every essential event B. Let $P^1 = \{A_1,\ldots,A_s\}$ be a partition with at least two essential events. Then there exist nonnegative $(p_j^1)_{j=1}^{s}$, summing to 1, and a continuous function $U^1 : C \to \mathbb{R}$, s.t. $\Sigma_{j=1}^{s} \alpha_j 1_{A_j} \succcurlyeq \Sigma_{j=1}^{s} \beta_j 1_{A_j} \Leftrightarrow \Sigma_{j=1}^{s} p_j^1 U^1(\alpha_j) \geq \Sigma_{j=1}^{s} p_j^1 U^1(\beta_j)$. The p_j^1's are uniquely determined, and U^1 is "cardinal", i.e. can be replaced by $\tilde{U}$ iff real τ and positive σ exist s.t. $\tilde{U} = \tau + \sigma U^1$.

PROOF. Define $\succcurlyeq'$ on C^n by $(\alpha_j)_{j=1}^{n} \succcurlyeq' (\beta_j)_{j=1}^{n}$ if $\Sigma_{j=1}^{n} \alpha_j 1_{A_j} \succcurlyeq \Sigma_{j=1}^{n} \beta_j 1_{A_j}$. Then $\succcurlyeq'$ satisfies all requirements of Theorem 5.1.(ii) of Wakker (1984b), the extension of Theorem 3.1 of Wakker (1984a). This then yields the first result of the above Lemma. The uniqueness results are standard, and can be obtained by adaption of those in and after Theorem 2.1 of Wakker (1984a). Let us emphasize that no topological separability is needed here. This

is because the proof of Wakker (1984b) uses the strengthenings in Chapter 6 of Krantz et al. (1971) of Theorems of Debreu (1960) and Blaschke and Boll (1938), leaving out topological separability.

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LEMMA 4.3. Let, under the assumptions and notation of Lemma 4.2, $P^2 = \{((A_{j_k})_{k=1}^{t_j})_{j=1}^{s}\}$ be a partition s.t. $A_j = \cup_{k=1}^{t_j} A_{j_k}$ for all j, so P^2 is finer than P^1. Then P^2 also has at least two essential events. The nonnegative $((P_{j_k})_{k=1}^{t_j})_{j=1}^{s}$, summing to one, and U^2, yielded by application of Lemma 4.2 to P^2, are such that $p_j^1 = \Sigma_{k=1}^{t_j} p_{j_k}$ for all j, and $U^2 = \tau + \sigma U^1$ for real τ and positive σ.

PROOF. If e.g. A_1 and A_2 are essential, then both $(A_{1k})_{k=1}^{t_1}$, and $(A_{2k})_{k=1}^{t_2}$, must contain at least one essential event. So we can indeed apply Lemma 4.2 to P^2, yielding $((p_{jk})_{k=1}^{t_j})_{j=1}^{s}$ and U_2. But now, defining $p_j := \Sigma_{k=1}^{t_j} p_{jk}$ for all j, and $U = U_2$, we get an array $(p_j)_{j=1}^{n}$ and an U, that satisfy the "$\Leftrightarrow$" requirement for $(p_j^1)_{j=1}^{n}$ and U^1 in Lemma 4.2. Hence by the uniqueness result of Lemma 4.2 we get $p_j^1 = p_j$ for all j, and $(\tilde{U}=)\ U_2 = \tau + \sigma U_1$ for real τ and positive σ.

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LEMMA 4.4. Let, under the assumptions of Lemma 4.2, $P^3 = \{B_1,\ldots,B_t\}$ be a partition with at least two essential events. Let application of Lemma 4.2 give $(p_j^3)_{j=1}^{t}$ and U^3. Then $U^3 = \tau + \sigma U^1$ for real τ and positive σ, and if $B_1 = A_1$ then $p_1^1 = p_1^3$.

PROOF. Define $P^2 = ((A_j \cap B_k)_{k=1}^{t})_{j=1}^{s}$, and apply Lemma 4.3. ∎

THEOREM 4.1. *Let C be topologically separable if I is essential and no two disjoint essential events exist. Then are equivalent:*

(i) There exist a finitely additive probability measure P, and a continuous function $U : C \to \mathbb{R}$, s.t. $\Sigma_{j=1}^{s} \alpha_j 1_{A_j} \succcurlyeq \Sigma_{k=1}^{t} \beta_k 1_{B_k}$ $\Leftrightarrow \Sigma_{j=1}^{s} P(A_j)U(\alpha_j) \geq \Sigma_{k=1}^{t} P(B_k)U(\beta_k)$ for all $\alpha_1,\ldots,\beta_t$.

(ii) $\succcurlyeq$ *is a simply-continuous weak order on* F^S, *and every event is CCI related to every essential event.*

Furthermore, if (i) applies, then we have the following uniqueness results:

(1) If I is inessential, then P is arbitrary and U can be any constant function.

(2) If I is essential, but no two disjoint essential events exist, then P assigns 1 to every essential event, 0 to every inessential event, and then U can be replaced by continuous $\tilde{U}$ iff $\tilde{U} = \phi \circ U$ for some strictly increasing ϕ.

(3) If two disjoint essential events exist, then P is uniquely determined, and U can be replaced by $\tilde{U}$ iff $\tilde{U} = \tau + \sigma U$ for real τ and positive σ.

PROOF. As always, (i) $\Rightarrow$ (ii) is direct, so we assume (ii), and derive (i), and the uniqueness results. For the case (1), I inessential, everything is straightforward. The case (2) can be handled by Lemma 4.1. So throughout this proof we assume we are in case (3). There exist two disjoint essential events A and A', thus A and A^c are essential. Any partition, finer than $\{A,A^c\}$, now has at least two essential events, one contained in A, one in A^c. For any event B we get a unique probability $P(B) = P(B \cap A) + P(B \cap A^c)$ from application of Lemma 4.2 to $P^1 = \{A \cap B, A \cap B^c, A^c \cap B, A^c \cap B^c\}$. Applying Lemma 4.2 to $\{A,A^c\}$ gives a utility function U. Now all desired results follow from Lemmas 4.3 and 4.4, but for brevity we do not spell out the details.

DEFINITION 4.1. We say $\succcurlyeq$ is *Coordinate Independent (CI)* on $F' \subset C^I$ if $x \succcurlyeq y \Rightarrow v \succcurlyeq w$ whenever $x,y,v,w \in F'$, and an event A exists such that $x_i = y_i$, $v_i = w_i$ on A, $x_i = v_i$, $y_i = w_i$ on A^c.

This property is known under various names as "independence", "sure-thing principle", "separability".

LEMMA 4.5. Let $\succcurlyeq$ satisfy the assumptions of Theorem 4.1, and (i) and (ii) there. Then $\succcurlyeq$ is CI on F^s, and $x \succcurlyeq y$ for all $x,y \in F^s$ with $\overline{x_i} \succcurlyeq \overline{y_i}$ for all $i \in I$.

PROOF. Straightforward.

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5. RESULTS FOR GENERAL ALTERNATIVES

In this section we want to extend the representation of Theorem 4.1.(i) to more general alternatives, mainly those of F^b, the set of all strongly bounded alternatives x (there are α, β such that $\bar{\alpha} \preccurlyeq \bar{x}_i \preccurlyeq \bar{\beta}$ for all i). We have in mind an expected utility representation by means of some sort of integral of U w.r.t. P. The approach to integration w.r.t. only finitely additive measures of section I.III.2 of Dunford and Schwarz (1958) or section 4.4 of Bhaskara Rao and Bhaskara Rao (1983) is not well suited for our purposes because we see no easy way to reformulate their properties of P and $U \circ x$ in terms of our primitive, i.e. $\succcurlyeq$. The less general Stieltjes type approach, as exposed in section 4.5 of Bhaskara Rao and Baskara Rao (1983) does serve our purposes. In this, an integral, notation EU, of a bounded measurable function $U \circ x$ on I is obtained as a "lower intergral", equal to $\sup \{EU(f^s) \mid f^s : I \to \mathbb{R} \text{ simple}, f^s \preccurlyeq^p U \circ x\}$, with $\preccurlyeq^p$ pointwise dominance, i.e. $f^s \preccurlyeq^p U \circ x$ whenever $f^s(i) \leq U(x_i)$ for all $i \in I$; or as an upper integral, which is analogous and yields the same result for bounded functions. If $U \circ x$ is bounded below (above) but unbounded above (below), one may still define the lower (upper) integral, and see if this is useful. Of course, we have in mind to let the f^s above be of the form $U \circ x^s$ for $x^s \in F^s$. We handle pointwise dominance as follows:

DEFINITION 5.1. We say $\succcurlyeq$ is *pointwise monotone* on $F' \subset C^I$ if $x \succcurlyeq y$ for all $x,y \in F'$ for which $\overline{x_i} \succcurlyeq \overline{y_i}$ for all $i \in I$.

This property is analogous to A9 of Suppes (1956), C1 of Ferreira (1972), and section IV.2 of Toulet (1984). An example to illustrate that pointwise monotonicity cannot be left out in Theorem 5.1 below, is the case where $I =]0,1]$, $C = \mathbb{R}$, A Borel σ-algebra, U is identity, P is Lebesgue measure, and $\succcurlyeq$ is represented by a linear functional V from F to $\mathbb{R}$, with $V(1_A) = P(A)$ for all events A. Then $\succcurlyeq$ is a constant- and simply-continuous weak order that even is CI. Every event is CCI related to every essential event . Yet, without pointwise monotonicity we are still completely free to let V assign to x, with $x_i = i$ for all i, any real number, such as -1.

LEMMA 5.1. If $\succcurlyeq$ is a constant-continuous pointwise monotone weak order on

F^b, then for every $x \in F^b$ there exists $\alpha \in C$ such that $x \approx \bar{\alpha}$.

PROOF. $\{\beta \in C \mid \bar{\beta} \succcurlyeq x\}$ and $\{\beta \in C \mid \bar{\beta} \preccurlyeq x\}$ are closed by constant-continuity, and nonempty if $x \in F^b$, because then [$\bar{\mu} \succcurlyeq \overline{x_i} \succcurlyeq \bar{\nu}$ for all i] and pointwise monotonicity imply μ to be in the first, ν in the second, set above. These sets have nonempty intersection by connectedness of C. Let α be in this intersection.

∎

We can now simply, for $x \in F^b$, take α as above, and define $EU(x) := U(\alpha)$, with U as in Theorem 4.1. Then $x \succcurlyeq y \iff EU(x) \geq EU(y)$. For $x = \Sigma_{j=1}^{m} \alpha_j 1_{A_j}$, $EU(x) = \Sigma_{j=1}^{m} P(A_j)U(\alpha_j)$. Beyond we shall see that the function EU can be considered an integral on all F^b.

THEOREM 5.1. Let C be topologically separable if I is essential, but no two disjoint essential events exist. Let $\succcurlyeq$ be a simply- and constant-continuous, and pointwise monotone, weak order on F^b, such that every event is CCI related to every essential event. Then there exist a finitely additive probability measure P, *and a continuous $U : C \to \mathbb{R}$, such that for all $x,y \in F^b$, $x \succcurlyeq y \iff \int U(x_i)dP(i) \geq \int U(y_i)dP(i)$, with these integrals well defined.*

PROOF. Let P,U be as yielded by Theorem 4.1. Let $x \in F^b$, $\bar{\mu} \succcurlyeq \overline{x_i} \succcurlyeq \bar{\nu}$ for all $i \in I$. Let $\bar{\alpha} \approx x$ (Lemma 5.1) and $EU(x) = U(\alpha)$. We must show that EU can be considered an integral. If $\bar{\mu} \approx \bar{\nu}$, then by pointwise monotonicity $x \approx \bar{\mu}$, so $\bar{\alpha} \approx \bar{\mu}$, and everything follows. So suppose $\bar{\mu} \succ \bar{\nu}$. For notational convenience we shall suppose that $U(\mu) = 1$, $U(\nu) = 0$. We now construct a sequence of pairs of simple functions $(x^m,y^m)_{m=1}^{\infty}$, s.t.
$U(x_i) - 1/m \leq U(x_i^m) \leq U(x_i) \leq U(y_i^m) \leq U(x_i) + 1/m$ for all i,m. For any m, and $0 \leq k \leq m-1$, $A_k^m := \{i \in I \mid k/m \leq U(x_i) < (k+1)/m\}$ is an event. Since U is continuous, and C connected, also $U(C) \subset \mathbb{R}$ is connected, so for any $0 \leq k \leq m$ there exists α_k s.t. $U(\alpha_k) = k/m$. We define

$$x^m := \Sigma_{k=0}^{m-1} \alpha_k 1_{A_k^m} + \alpha_{m-1} 1_{\{i \mid U(x_i)=1\}}, \text{ and}$$

$$y^m := \Sigma_{k=0}^{m-1} \alpha_{k+1} 1_{A_k^m} + \alpha_m 1_{\{i \mid U(x_i)=1\}}.$$

We then have $U(x_i^m) \leq U(x_i) \leq U(y_i^m)$, so $\overline{x_i^m} \preccurlyeq \overline{x_i} \preccurlyeq \overline{y_i^m}$, for all i. By point-

wise monotonicity $x^m \preccurlyeq x \preccurlyeq y^m$. Thus $EU(x^m) \leq U(\alpha) \leq EU(y^m)$. But also $EU(y^m) - EU(x^m) = 1/m$ for all m. We conclude that $EU(x) = U(\alpha) = \lim_{m\to\infty} EU(x^m) = \lim_{m\to\infty} EU(y^m)$. And thus indeed EU(x) can be considered to be an integral of U w.r.t. P.

∎

For a not strongly bounded alternative x the case is not so easy. Of course, if yet $x \approx \bar{\alpha}$ for some $\alpha \in C$, which always occurs if $\bar{\mu} \succcurlyeq x \succcurlyeq \bar{\nu}$ for some $\mu,\nu \in C$, we would like still to define $EU(x) = U(\alpha)$. But now there is no justification to consider this as an integral of $U \circ x$. If x is strongly bounded below (there is γ such that $\overline{x_i} \succcurlyeq \bar{\gamma}$ for all i) an integral value for $U \circ x$, its "lower integral", exists. This limit is not greater than EU(x), may equal EU(x), but may also very well be smaller than EU(x). If $x \approx \bar{\alpha}$, but now x strongly bounded above, and not below, we can obtain an upper integral value that may be "too" large.

Conditions for $\succcurlyeq$, strong enough to guarantee that $\succcurlyeq$ can be represented by an integral for all acts, usually are undesirably strong, e.g. they may simply imply boundedness of U, as turned out to be the case in Savage (1954). Or they may even lead to impossibility results, e.g. if $C =]0,1] = I$, $\succcurlyeq$ maximizes Lebesgue integral, and one would let $A = 2^I$ and require continuity of $\succcurlyeq$ w.r.t. the product topology on C^I. Then this would require a σ-additive extension of Lebesgue measure to $2^{]0,1]}$ which is known not to exist. Or, finally, such conditions may restrict the set of considered alternatives strongly.

The integral representation càn be extended to those alternatives x, equivalent to some $\bar{\alpha}$, that have a "good enough" consequence γ for every $\bar{\beta} \prec x$, to ensure that the "above truncation" x' of x at γ (i.e. $x_i' = x_i$ if $\overline{x_i} \preccurlyeq \bar{\gamma}$, $x_i' = \gamma$ if $\overline{x_i} \succ \bar{\gamma}$) has $\bar{\beta} \prec x'$, and that have a "bad enough" consequence ν for every $\bar{\mu} \succ x$, to ensure that the "below truncation" x" of x at ν (i.e. $x_i'' = x_i$ if $\overline{x_i} \succcurlyeq \bar{\nu}$, $x_i'' = \nu$ if $\overline{x_i} \prec \bar{\nu}$) has $\bar{\mu} \succ x''$. This is the way to extend $\succcurlyeq$ to the class of all alternatives with finite expected utility, a desirable result for instance for statistical applications (see De Groot, 1970, end of §7.9). For brevity we do not elaborate that here.

Other alternatives are difficult to handle. One quickly gets problems with variations of the St. Petersburg paradox. If one adds preferences between these alternatives by means of CI and pointwise dominance, then a problem may be how to preserve other desirable matters, such as transitivity. These matters are addressed in Toulet (1984).

6. COUNTABLE ADDITIVITY

We shall give a continuity assumption, necessary and sufficient for σ-additivity of the probability measure P of Theorems 4.1 and 5.1. This adapts the known results, as presented in section 6.9 of the Finetti (1972) to the more general case where $C \neq \mathbb{R}$, in terms of a preference relation $\succcurlyeq$. Property F_7 in section 10.3 of Fishburn (1982), and the "monotone continuity" assumption of Villegas (1964), Arrow (1970, Lecture 1) are analogous.

DEFINITION 6.1. A probability measure P on a field A is *countably* (or σ-) *additive* if $\lim_{m\to\infty} P(A_m) = 0$ whenever $A_{m+1} \subset A_m$ for all m, and $\cap_{m=1}^{\infty} A_m = \phi$, for any sequence of events $(A_m)_{m=1}^{\infty}$.

It is known that this applies iff $P(B) = \Sigma_{m=1}^{\infty} P(B_m)$ whenever the B_m's are mutually disjoint, and their union $B = \cup_{m=1}^{\infty} B_m$ happens to be in the field A.

DEFINITION 6.2. We say $\succcurlyeq$ is *boundedly strictly continuous* if for any sequence of alternatives $(x^j)_{j=1}^{\infty}$, $[x \succ y$ (respectively $x \prec y)]$ follows from $[x^j \succ y$ (respectively $x^j \prec y)$ for all j, and $\lim_{j\to\infty} x_i^j = x_i$ for all i $]$, whenever $(x^j)_{j=1}^{\infty}$ is uniformly strongly bounded, i.e. μ and ν exist such that $\bar{\mu} \succcurlyeq \bar{x}_i^j \succcurlyeq \bar{\nu}$ for all $j \in \mathbb{N}$, $i \in I$.

Observe that the above definition is a weakened version of continuity w.r.t. the product topology, i.e. w.r.t. pointwise convergence. For one thing, we only consider (denumerable) sequences. Furthermore there is the restriction of uniform strong boundedness.

THEOREM 6.1. Let the conditions of Theorem 5.1 be satisfied. Then P can be chosen σ-additive iff $\succcurlyeq$ is boundedly strictly continuous.

PROOF. First we assume bounded strict continuity, and derive σ-additivity. If I is inessential we have U constant, and can take any σ-additive P, e.g. P(A) = 1 iff A contains some fixed $i \in I$.

So suppose I essential. Then α,β exist s.t. $\bar{\alpha} \succ \bar{\beta}$, otherwhise pointwise monotonicity, or CCI, would imply inessentiality of I. Now let $A_m \supset A_{m+1}$ for all m, $\cap A_m = \phi$. Define $x^m = \alpha 1_{A_m} + \beta 1_{A_m^c}$, $x = \bar{\beta}$. By pointwise monotonicity $x^m \succcurlyeq x^{m+1} \succcurlyeq \bar{\beta}$ for all m, so $\lim_{m\to\infty} EU(x^m) \geq U(\beta)$. We have $\lim_{m\to\infty} x_i^m = x_i$ for all i, and $(x^m)_{m=1}^{\infty}$ is uniformly strongly bounded. Hence $x^m \succ \bar{\gamma}$ for all m implies $x = \bar{\beta} \succ \bar{\gamma}$, for any γ. Since U(C) is connected, for any $0 < \varepsilon \leq U(\alpha) - U(\beta)$ there is γ with $U(\gamma) = U(\beta) + \varepsilon$. Now $\lim_{m\to\infty} EU(x^m) > U(\beta)$ would imply existence of γ with $x^m \succ \bar{\gamma}$ for all m, and $U(\gamma) > U(\beta)$. Now $x^m \succ \bar{\gamma}$ for all m implies $\bar{\beta} = x \succ \bar{\gamma}$, contradicting $U(\gamma) > U(\beta)$. It follows that $\lim_{m\to\infty} EU(x^m) = U(\beta)$. From $EU(x^m) = P(A_m)U(\alpha) + (1-P(A_m))U(\beta)$ follows $\lim_{m\to\infty} P(A_m) = 0$, as required for σ-additivity of P.

Reversedly, let P be σ-additive. Then bounded strict continuity follows from continuity of U and the dominated convergence Theorem of Lebesgue (e.g. see Corollary 16 in §I.III.6.16 of Dunford and Schwarz (1958)). This theorem is usually formulated for σ-fields. It can be applied to our context by taking the smallest σ-field containing A, and taking the unique σ-additive extension of P to this, guaranteed by Royden (1963, §12.2). The values of the involved integrals of $U \circ x_m$ and $U \circ x$ are unaffected by this extension of A.

∎

7. CONCLUSION

First we formulate our main result. For clarity we repeat the assumptions made.

THEOREM 7.1. Let I be a nonempty set, A a field on I. Let C be a connected topological space, with a field $\mathcal{D}$ on it that contains all open and one-point sets. Let $F \subset C^I$ be the set of all alternatives that are A - $\mathcal{D}$ measurable. Let F^b be the subset of all strongly bounded (end section 2) *alternatives in* F. *Let* $\succ$ *be a binary relation on* C^I. *Let* C *be topologically separable if* I *is essential and no two disjoint essential events in* A *exist. Then are equivalent:*

(i) There exists a finitely additive probability measure P *on* A, *and a continuous* $U : C \to \mathbb{R}$, *such that* $x \succ y \iff \int U(x_i)dP(i) \geq \int U(y_i)dP(i)$, *integrals well defined, for all* $x,y \in F$ *for which there are* $\alpha,\beta,\mu,\nu \in C$ *such that* $U(\alpha) \leq U(x_i) \leq U(\beta)$ *and* $U(\mu) \leq U(y_i) \leq U(\nu)$ *for all* i.

(ii) $\succ$ *is a simply- and constant-continuous pointwise monotone weak order on* F^b, *for which all events are CCI related to all essential events.*

Furthermore, in (i) we may replace "finitely" by "countably" if in (ii) we require that $\succ$ *is boundedly strictly continuous.*

Uniqueness results are as in Theorem 4.1.

PROOF. See Theorems 4.1, 5.1, 6.1. ∎

To our knowledge this is the first characterization of expected utility maximization with continuous utility of this generality.

The special case where $C = \mathbb{R}$, and U identity, is treated in de Finetti (1972), a major source of inspiration for this paper.

The application of our result is not restricted to DMUU. The i's may refer to other things than states of nature. The CCI assumption does require the consequences at different i's to be appreciated, via the utility function U, in analogous ways. Hence the i's should make this possible. For instance one may think of the case where I is a set, possibly a continuum, of agents, who appreciate the consequences the same way; alternatives are allocations of the consequences over the agents, and P symbolizes power of the agents.

Another major source of applications lies in dynamic contexts, where the i's refer to time points, I being [0,1], $\mathbb{R}_+$, $\mathbb{N}$, or whatever; our analysis is general enough to apply to discrete and continuous time. Examples are dynamic programming, economic growth theory, the study of optimal consumption-saving behaviour of consumers. Here the alternatives are production or consumption paths in time. The function U is a time-independent instantaneous utility function. P denotes a (subjective, time-dependent) discount factor. Often the discount factor is assumed to be

constant, corresponding to weights of a form $k.e^{-\rho t}$. This can be characterized by an extra stationarity assumption. For applications, and further references, to dynamic contexts, see Shapley (1953), Yaari (1964), Koopmans (1972), Mirrlees (1974), Bailey, Olson and Wonnacott (1980), Drèze (1982), who emphasizes the analogy between the "I = states" and " I = time" interpretations, Denardo (1982), and Vrieze (1983).

If one takes such models as a starting point, and searches for further derivations of theoretical results, representation theorems like ours are not needed. But if one searches for practical applications, then one faces for instance the problem that the discount factor, or the probability distribution, are not given beforehand, but are subjective and unknown, and payment is not in utility, but in a real quantity as money, which is a transform of utility by an unknown (subjective) transformation. And then for justifying, or criticizing, the assumption of the existence of models as above, representation theorems like ours, and Koopmans (1972), (and many others) are needed. See also section III.1 of Koopmans (1972).

REFERENCES

ARROW, K.J. (1971): *Essays in the Theory of Risk-Bearing*. North-Holland, Amsterdam.

BAILEY, M.J., M. OLSON & P. WONNACOTT (1980): Marginal Utility of Income does not Increase: Borrowing, Lending and Friedman-Savage Gambles. *American Economic Review* 70, 372-379.

BHASKARA RAO, K.P.S. & M. BHASKARA RAO (1983): *Theory of Charges*. Academic Press, London.

BLASCHKE, W. & G. BOL (1938): *Geometrie der Gewebe*. Springer, Berlin.

DEBREU, G. (1959): *Theory of Value*. Wiley, New York.

DEBREU, G. (1960): Topological Methods in Cardinal Utility Theory. In K.J. Arrow, S. Karlin, & P. Suppes (Eds.), *Mathematical Methods in the Social Sciences*, 1959. Stanford University Press, Stanford, 16-26.

DE FINETTI, B. (1972): *Probability, Induction and Statistics*. Wiley, New York.

DEGROOT, M.H. (1970): *Optimal Statistical Decisions*. McGraw-Hill, New York.

DENARDO, E.V. (1982): *Dynamic Programming, Models and Applications*. Prentice-Hall, Englewood Cliffs, N.J.

DREZE, J.H. (1982): The Marginal Utility of Income Does Not Increase. Comment. *Core Discussion Paper* 8231. CORE, Louvain-la-Neuve.

DUNFORD, N. & J.T. SCHWARTZ (1958): *Linear Operators, Part I*. Interscience Publishers, New York.

FERREIRA, P.E. (1972): On Subjective Probability and Expected Utilities. *Annals of Mathematical Statistics* 43, 928-933.

FISHBURN, P.C. (1970): *Utility Theory for Decision Making*. Wiley, New York.

FISHBURN, P.C. (1981): Subjective Expected Utility: A Review of Normative Theories. *Theory and Decision* 13, 139-199.

FISHBURN, P.C. (1982): *The Foundations of Expected Utility*. Reidel, Dordrecht.

KELLEY, J.L. (1955): *General Topology*. Van Nostrand, London.

KOOPMANS, T.C. (1972): Representation of Preference Orderings over Time. In C.B. McGuire & R. Radner (Eds.), *Decision and Organization*. North-Holland, Amsterdam, 79-100.

KRANTZ, D.H., R.D. LUCE, P. SUPPES & A. TVERSKY (1971): *Foundations of

Measurement, *Vol*. I. Academic Press, New York.

MIRRLEES, J. (1974): Optimum Accumulation under Uncertainty: The Case of Stationary Returns to Investment. In J.H. Drêze (Ed.), *Allocation under Uncertainty, Equilibrium and Optimality*, MacMillan, New York.

ROYDEN, H.L. (1963): *Real Analysis*. MacMillan, New York.

SAVAGE, L.J. (1954): *Foundations of Statistics*. Wiley, New York.

SCHOEMAKER, P.J.H. (1982): The Expected Utility Model: Its Variants, Purposes, Evidence and Limitations. *Journal of Economic Literature* 20, 529-563.

SHAPLEY, L.S. (1953): Stochastic Games. *Proceedings of the National Academy of Sciences of the USA* 39, 1095-1100.

SUPPES, P. (1956): The Role of Subjective Probability and Utility in Decision Making. *Proceedings of the Third Berkeley Symposium on Mathematicsl Statistics and Probability*, 1954-1955, 5, 61-73.

TOULET, C. (1984): An Axiomatic Model of Unbounded Utility Functions, Université de Limoges, Laboratoire d'Econometrie de Paris VI. Paper presented at FUR-84 Conference.

VILLEGAS, C. (1964): On Quantitative Probability σ-Algebras. *Annals of Mathematical Statistics* 35, 1787-1796.

VRIEZE, O.J. (1983): *Stochastic Games with Finite State and Action Spaces*. Ph.D. Thesis, Mathematical Centre, Amsterdam.

WAKKER, P.P. (1984a): Cardinal Coordinate Independence for Expected Utility. *Journal of Mathematical Psychology*, forthcoming.

WAKKER, P.P. (1984b): A Characterization of Concave Additively Decomposable Representing Functions and Expected Utility with Risk Aversion. Report 8419, Department of Mathematics, University of Nijmegen.

YAARI, M.E. (1964): On the Consumer's Lifetime Allocation Process. *International Economic Review* 5, 304-317.

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