

Cardinal Coordinate Independence for Expected Utility

PETER WAKKER

University Nijmegen

A representation theorem for binary relations on \mathbb{R}^n is derived. It is interpreted in the context of decision making under uncertainty. There we consider the *existence* of a subjective expected utility model to represent a preference relation of a person on the set of bets for money on a finite state space. The theorem shows that, for this model to exist, it is not only necessary (as has often been observed), but it also is sufficient, that the appreciation for money of the person has a cardinal character, independent of the state of nature. This condition of cardinal appreciation is simple and thus easily testable in experiments. Also it may be of help in relating the neo-classical economic interpretation of cardinal utility to the von Neumann-Morgenstern interpretation.

1. INTRODUCTION

In this paper a characterization of the subjective expected utility (SEU) model is given. To this end in Section 2 a representation theorem for binary relations on \mathbb{R}^n is given. It is based mainly on Debreu (1960), or Chapter 6 of Krantz, Luce, Suppes, and Tversky (1971). The new characterizing property, denoted by CCI, is simple, and yields a handy, empirically testable, criterion. The question what such a property is, has more or less been posed (in the footnote on p. 19 of Marschak and Radner (1972), Section 7.2 of Fishburn (1970), and as Formula 2.7 of Drèze (1982). It leads to a cardinal utility function in the von Neumann-Morgenstern sense; i.e., its stochastic expected value is used to value bets. It may also appeal to the neo-classical economic interpretation, where cardinal utility is a psychological primitive, indicating strength of preference.

In Sarin (1982) and Fishburn (1970, Theorem 7.4) a quaternary relation \succcurlyeq^* , comparing preference differences, is introduced as a primitive. Then, in terms of \succcurlyeq^* , conditions are given under which a SEU model exists, such that it agrees with \succcurlyeq^* . We take as the point of departure that only the preference relation \succcurlyeq on the set of acts is observable. (In Section 3 we use the term "bets" instead of "acts.") A strength of preference relation \succcurlyeq^* may exist, but is not considered directly observable. Therefore our characterization is directly in terms of \succcurlyeq . With the notion of strength of preference in mind, one will not be surprised by the form of our new characterizing

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property. It guarantees the *possibility* of deriving from \succsim a strength of preference relation for money, independent of the state of nature. Thus the condition is not only necessary for the existence of a SEU model (as has often been observed), but it also is sufficient, in the present context.

Most common, in characterizations of SEU, is the approach, originating from von Neumann and Morgenstern (1944, 1947, 1953, Section I.3 and appendix), where lotteries on the set of acts are introduced. See, for instance, Fishburn (1982). If act x is preferred to a lottery that with probability $1/2$ yields act y , and with probability $1/2$ act z , then the strength of preference between x and y should exceed that between z and x . In Segal (1982) the case is considered where there is an upper bound for the number of prizes of lotteries.

In Camacho (1980) a preference relation on sequences of acts (maintaining terminology) is considered. If twice act x is preferred to once act y and once act z , then the strength of preference between x and y should exceed that between z and x . Thus in both mentioned approaches strength of preference directly derives from an extra notion.

In de Finetti (1937) the SEU model is derived, with the (implicit) assumption that the bets, considered there, can be summed. There, if bet x is preferred to y , then $x + v$ should be preferred to $y + v$ for all v . Then again, if $x + y$ is preferred to $z + w$, then the strength of preference between x and z exceeds that between w and y . (This will come down to the special case of our Theorem 3.1 where U is identity.)

The best known derivation of the SEU model is in the first five chapters of Savage (1954). Here only the preference relation \succsim on the set of acts is taken as primitive, and no strength of preference relation is brought in "from outside" by some extra notion. This has also been our aim. (Contrary to what is often thought, in Savage's model the state space can be denumerable. For other misunderstandings w.r.t. Savage (1954), see Wakker (1981).)

For surveys on SEU characterizations see Fishburn (1981) or Schoemaker (1982).

2. A REPRESENTATION THEOREM

This section will be technical. First some notations and definitions are introduced. Elements of \mathbb{R}^n are denoted by a, v, x , etc., with coordinates a_1, x_2 , etc. Elements of \mathbb{R} are denoted by Greek characters, α, β , etc. (\emptyset excepted). A notation that is less standard is the following: $(x_{-i}\alpha)$ is the element of \mathbb{R}^n with i th coordinate α , other coordinates equal to those of x ; $(x_{-i,j}\alpha, \beta)$ is the element of \mathbb{R}^n with i th coordinate α , j th coordinate β , other coordinates equal to those of x . For a binary relation \succsim we write $x \succsim y$ instead of $(x, y) \in \succsim$. For the usual properties of binary relations the reader is referred to the references. Some more notation, $x \leq y$ if $y \succsim x$, $x < y$ if not $x \succsim y$, $x > y$ if not $x \leq y$, $x \approx y$ if $x \succsim y$ and $y \succsim x$. A *weak order* \succsim is total ($x \succsim y$ or $y \succsim x$ for all x, y) and transitive, inducing an equivalence relation \approx . Coordinate i is *essential* (w.r.t. \succsim) if $(x_{-i}\alpha) > (x_{-i}\beta)$ for some x, α, β . We say \succsim is *monotone* if $x \succsim y$ whenever $x_j \succsim y_j$ for all j , and \succsim is continuous if $\{x \mid x \succsim y\}$ and $\{x \mid x \leq y\}$ are closed for all y .

DEFINITION. We say \succcurlyeq is *coordinate independent (CI)* if

$$(x_{-i}\alpha) \succcurlyeq (y_{-i}\alpha) \Rightarrow (x_{-i}\beta) \succcurlyeq (y_{-i}\beta) \text{ for all } x, y, i, \alpha, \beta.$$

DEFINITION. We say \succcurlyeq is *cardinally coordinate independent (CCI)* if for all $x, y, v, w, \alpha, \beta, \gamma, \delta, j$, and essential i , from $(x_{-i}\alpha) \preccurlyeq (y_{-i}\beta)$ & $(x_{-i}\gamma) \succcurlyeq (y_{-i}\delta)$ & $(v_{-j}\alpha) \succcurlyeq (w_{-j}\beta)$ follows $(v_{-j}\gamma) \succcurlyeq (w_{-j}\delta)$.

Elucidation. Replacement of (α, β) by (γ, δ) changes \preccurlyeq into \succcurlyeq , from the first preference to the second. We imagine this replacement should thus kind of “reinforce” \succcurlyeq from the third to the fourth preference. So certainly not should it change \succcurlyeq into \prec , from the third to the fourth preference.

DEFINITION. We say \succcurlyeq is *standard sequence invariant* if for all $x, y, \mu, v, \sigma, \tau$, $i \neq k, j \neq l$, with i essential, the relation $(y_{-j,l}\beta, \sigma) \approx (y_{-j,l}\gamma, \tau)$ is implied by the relations $(x_{-i,k}\alpha, \mu) \approx (x_{-i,k}\beta, v)$, $(x_{-i,k}\beta, \mu) \approx (x_{-i,k}\gamma, v)$, and $(y_{-j,l}\alpha, \sigma) \approx (y_{-j,l}\beta, \tau)$.

This property is (a minor variation on) the condition in Theorem 15(i) of Section 6.11.2 of Krantz *et al.* (1971). Their Axiom 5 in Section 8.2.6, formulated in a complex structure, comes closest to our CCI.

For the case $n = 2$ we shall need one more property.

DEFINITION. If $n = 2$, then \succcurlyeq satisfies the *Thomsen–Blaschke (TB) condition* if the relation $(\gamma, \mu) \approx (\alpha, v)$ follows from $(\alpha, \mu) \approx (\beta, v)$, $(\alpha, v) \approx (\beta, \sigma)$, and $(\gamma, v) \approx (\alpha, \sigma)$, for all α, \dots, σ .

LEMMA 2.1. *If \approx is an equivalence relation, then $x \approx y$ whenever $x_j = y_j$ for all essential j .*

Proof. As an example, let 1, 2, 3 be not essential, $x_j = y_j$ for all $j \geq 4$. Then $x \approx (x_{-1}y_1) \approx (x_{-1,2}y_1, y_2) \approx ((x_{-1,2}y_1, y_2)_{-3}y_3) = y$. ■

LEMMA 2.2. *If \approx is an equivalence relation, then CCI implies CI.*

Proof. If no coordinate is essential this follows from Lemma 2.1. Otherwise reflexivity of \succcurlyeq is sufficient. To see this, let $\beta \in \mathbb{R}$, $(x_{-j}\alpha) \succcurlyeq (y_{-j}\alpha)$. Let i be essential. Then apply CCI with $(v_{-i}\alpha) \preccurlyeq (v_{-i}\alpha)$, $(v_{-i}\beta) \succcurlyeq (v_{-i}\beta)$, $(x_{-j}\alpha) \succcurlyeq (y_{-j}\alpha)$, yielding $(x_{-j}\beta) \succcurlyeq (y_{-j}\beta)$. Here v is arbitrary. ■

LEMMA 2.3. *CCI implies standard sequence invariance.*

Proof. It is logically stronger. ■

LEMMA 2.4. *If \approx is an equivalence relation, or if at least one coordinate is essential, then CCI implies TB. (For $n = 2$.)*

Proof. Let $n = 2$. If no coordinate is essential Lemma 2.1 applies. Next let coordinate 1 be essential. Let $i = j = 1$, $(\alpha, v) \leq (\beta, \sigma)$, $(\gamma, v) \geq (\alpha, \sigma)$, $(\alpha, \mu) \geq (\beta, v)$, then CCI yields $(\gamma, \mu) \geq (\alpha, v)$. Exchanging in these preferences first and second bet gives $(\beta, \sigma) \leq (\alpha, v)$, $(\alpha, \sigma) \geq (\gamma, v)$, $(\beta, v) \geq (\alpha, \mu)$, so $(\alpha, v) \geq (\gamma, \mu)$ by CCI. Thus CCI implies TB. If coordinate 2 is essential a symmetric argument applies. ■

We are now ready for our main theorem.

THEOREM 2.1. *Let \succcurlyeq be a binary relation on \mathbb{R}^n . Then the following (i) and (ii) are equivalent:*

(i) *There exist nonnegative $(\lambda_j)_{j=1}^n$ and a continuous function $U: \mathbb{R} \rightarrow \mathbb{R}$, such that $x \succcurlyeq y \Leftrightarrow \sum_{j=1}^n \lambda_j U(x_j) \geq \sum_{j=1}^n \lambda_j U(y_j)$ for all x, y .*

(ii) *\succcurlyeq is a continuous weak order that satisfies CCI.*

Furthermore, if (i) applies, and two or more coordinates are essential, then U is an interval scale (i.e., can be replaced by U' if and only if β and positive α exist such that $U' = \alpha U + \beta$).

Proof. It is straightforward that (i) implies (ii). Next we assume (ii), and derive (i). If no coordinate is essential, then by Lemma 2.1 we see that $x \approx y$ for all x, y ; and everything follows. If one coordinate, say i , is essential, then we apply Section 4.6 of Debreu (1959). This guarantees existence of a continuous function $\phi: \mathbb{R}^n \rightarrow \mathbb{R}$, such that for all x, y we have $x \succcurlyeq y \Leftrightarrow \phi(x) \geq \phi(y)$. Then let $\phi(x) = U(x_i)$, $\lambda_i = 1$, $\lambda_j = 0$ for all $j \neq i$.

If two or more coordinates are essential, then first we guarantee existence of continuous functions $(V_j)_{j=1}^n$, such that $[x \succcurlyeq y \Leftrightarrow \sum_{j=1}^n V_j(x_j) \geq \sum_{j=1}^n V_j(y_j)]$ for all $x, y \in \mathbb{R}^n$. If exactly two coordinates are essential, then we leave out the other coordinates. This may be done by Lemma 2.1; the V_j 's associated with the removed coordinates j are taken constant. As indicated in Debreu (1960), the "Hauptsatz über Sechseckgewebe" of Section 1.2 of Blaschke and Bol (1938) guarantees the existence of $(V_j)_{j=1}^n$ as above, for any continuous weak order on \mathbb{R}^2 that is CI and TB, and has both coordinates essential. CI and TB of \succcurlyeq have been guaranteed by our Lemmas 2.2 and 2.4.

If three or more coordinates are essential, then we apply Theorem 3 of Debreu (1960). This guarantees existence of $(V_j)_{j=1}^n$ as above, for any continuous weak order on \mathbb{R}^n that is CI, and has at least three coordinates essential. CI of \succcurlyeq has been guaranteed by our Lemma 2.2.

So, if two or more coordinates are essential, then $(V_j)_{j=1}^n$ as above exist. By Lemma 2.3 we see that \succcurlyeq is standard sequence invariant. By Theorem 15 of Section 6.11.2 of Krantz *et al.* (1971) this is (necessary and) sufficient for the existence of real numbers $(\lambda_j)_{j=1}^n$, and a continuous U , such that $V_j = \lambda_j U$ for all j . Since CCI excludes $(x_{-i}\alpha) \succ (x_{-i}\beta) \ \& \ (y_{-j}\beta) \succ (y_{-j}\alpha)$, all λ_j 's are either nonpositive (then replace U by $-U$, and λ_j by $-\lambda_j$), or nonnegative. Hence (i) is derived. ■

If (i) and (ii) are satisfied, one may furthermore note that, if no coordinate is essential, then $x \succcurlyeq y$ for all x, y by Lemma 2.1, and U should be chosen constant, or

all λ_j 's are chosen zero. If one or more coordinates are essential, then U is not constant, and λ_j is zero if and only if coordinate j is not essential, and $(\lambda_j)_{j=1}^n$ can be replaced by $(\lambda'_j)_{j=1}^n$ if and only if there is $\alpha > 0$ such that $\lambda'_j = \alpha\lambda_j$ for all j . If exactly one coordinate is essential, then U is ordinal, i.e., U may be replaced by U' if and only if there exists increasing ϕ such that $U = \phi \circ U'$ (U' continuous $\Leftrightarrow \phi$ continuous).

For the case where \succsim applies to a bounded subset of \mathbb{R}^n , the following result is of use.

THEOREM 2.2. *Let \succsim be a binary relation on $(\sigma, \tau)^n$, where $-\infty \leq \sigma < \tau \leq \infty$. Then the results of Theorem 2.1 apply (with now \succsim on $(\sigma, \tau)^n$, and $U: (\sigma, \tau) \rightarrow \mathbb{R}$).*

Proof. We introduce a bijective increasing (thus continuous) map $\phi: \mathbb{R} \rightarrow (\sigma, \tau)$, and define \succsim' on \mathbb{R}^n by $(x_1, \dots, x_n) \succsim' (y_1, \dots, y_n)$ if and only if $(\phi(x_1), \dots, \phi(x_n)) \succsim (\phi(y_1), \dots, \phi(y_n))$. Then we apply Theorem 2.1, and define U' on (σ, τ) by $U'(y') = U(y)$, where $y' = \phi(y)$, for all y' in (σ, τ) . The essential point in this is that the map ϕ does not affect transitivity, totality, continuity, or CCI, of \succsim . ■

3. DECISION MAKING UNDER UNCERTAINTY

In this section we interpret the previous theorems in the context of decision making under uncertainty. Let $S = \{s_1, \dots, s_n\}$ be a finite set, called the *state space*, where $n \in \mathbb{N}$. Its elements are (*possible states (of nature)*). Subsets of S are *events*. Exactly one is the *true state*; the other states are not true. A person T is uncertain which of the states is true. As an example, one may think of horse races. Of n participating horses exactly one will win. By s_j we denote the “possible state of nature” that horse j will win. An element $x = (x_1, \dots, x_n)$ of $(\sigma, \tau)^n$, where $-\infty \leq \sigma < \tau \leq \infty$, denotes a *bet* for money, that yields $\mathcal{S}x_j$ if s_j is the true state. Money is assumed to be a continuously divisible quantity that can take all real values between σ and τ (also negative, if $\sigma < 0$). By \succsim we denote the preference relation of T on the set of bets. (We use the term “bet” instead of “act” in this section.) If T prefers x to y we write $x \succ y$.

We assume in descriptive context, if T chooses x out of $\{x, y\}$, then $x \succ y$, and in prescriptive context, if $x \succ y$, then T will be willing to choose x out of $\{x, y\}$. We consider \succsim as the only observable entity in our model, the other entities have their meaning through \succsim .

DEFINITION. We say (SEU=) $[(\sigma, \tau)^n, \succsim, (p_j)_{j=1}^n, U]$ is a *Subjective Expected Utility (SEU) model* (for \succsim) if $-\infty \leq \sigma < \tau \leq \infty$, \succsim is a binary relation on $(\sigma, \tau)^n$, the p_j 's are nonnegative real numbers that sum to 1, and $U: (\sigma, \tau) \rightarrow \mathbb{R}$ is a function, such that $[x \succ y \Leftrightarrow \sum_{j=1}^n p_j U(x_j) \geq \sum_{j=1}^n p_j U(y_j)]$ for all x, y . Then p_j is the *subjective probability* for state s_j , U the (*subjective*) *utility function* (for money), and $\sum_{j=1}^n p_j U(x_j)$ the (*subjective*) *expected utility* for bet x (under SEU).

It is the applicability of the SEU models that we characterize in this section. We restrict our attention to the special case of continuous utility functions. One quickly observes that a necessary property for the appropriateness of an SEU model is that \succsim is CI. In the context of decision making under uncertainty CI is known under the name “Sure-thing Principle” (Savage, 1954). In Economics it is called something like “separability,” such as “strong separability” in Barten and Böhm (1982), “complete strict separability” in Blackorby, Primont, and Russel (1978), or “additivity” in Katzner (1970). In Measurement Theory it is called “independence,” such as in Debreu (1960), Krantz *et al.* (1971), and Roberts (1979), or “mutual preferential independence,” Keeney and Raiffa (1976). We like most the following verbal formulation for it: the preference between bets x and y depends only on (their values at) the states where they differ; it is independent of (their values at) states where they are equal.

To indicate the meaning of CCI for decision making under uncertainty, and to show how it is related to strength of preference, consider the following.

DEFINITION. We say $(\alpha, \beta) \succsim_i^*(\gamma, \delta)$ if there exist x, y such that $(x_{-i}\alpha) \succsim (y_{-i}\beta)$, and $(x_{-i}\gamma) \preccurlyeq (y_{-i}\delta)$; we also say $(\alpha, \beta) \succ_i^*(\gamma, \delta)$ if \succsim can be replaced by \succ , or \preccurlyeq by \prec .

For the idea of this definition, suppose $(x_{-i}\alpha) \succ (y_{-i}\beta)$, $(x_{-i}\gamma) \prec (y_{-i}\delta)$, and $\alpha > \beta, \gamma > \delta$. Then, since $\alpha > \beta$, the possibility of state s_i seems to yield a positive argument for bet $(x_{-i}\alpha)$ as compared to bet $(y_{-i}\beta)$, assuming T prefers more money to less money. If we now replace (α, β) by (γ, δ) , then this still applies. Yet by this the preference has reversed. Since at the other states of nature nothing has changed, this reversal can only be explained by the change of matters at state s_i . Apparently the positive argument, yielded by the possibility that state s_i is true, has lost some of its strength by the replacement. We conclude that at state s_i the *strength of preference* between (α, β) exceeds that between (γ, δ) , and write $(\alpha, \beta) \succ_i^*(\gamma, \delta)$. Note that this strength of preference notion has been derived from the preference relation \succsim .

Of course, in general the above definition does not have to make sense, the \succsim^* and \succ^* relations may have undesired properties.

DEFINITION. We say *the appreciation for money (of T) has a cardinal character, independent of the state of nature*, if $(\alpha, \beta) \succsim_i^*(\gamma, \delta) \ \& \ i \text{ essential} \Rightarrow \text{not } (\gamma, \delta) \succ_j^*(\alpha, \beta)$, for all $\alpha, \beta, \gamma, \delta, i, j$.

LEMMA 3.1. *The appreciation for money has a cardinal character, independent of the state of nature, if and only if \succsim is CCI.*

Proof. Straightforward. Note:

$$(\alpha, \beta) \succsim_i^*(\gamma, \delta) \Leftrightarrow (\delta, \gamma) \succsim_i^*(\beta, \alpha);$$

$$(\alpha, \beta) \succ_i^*(\gamma, \delta) \Leftrightarrow (\delta, \gamma) \succ_i^*(\beta, \alpha). \blacksquare$$

THEOREM 3.1. *There exists an SEU model with continuous U for \succsim , if and only if \succsim is a continuous weak order, and the appreciation for money of T has a cardinal character, independent of the state of nature. Then \succsim is CI; and U is non-decreasing if and only if \succsim is monotone.*

Proof. The first assertion follows from Lemma 3.1 and Theorem 2.2. The second is straightforward. ($\sum_{j=1}^n p_j = 1$ can always be arranged.) ■

In descriptive context, this characterization shows that, taking the continuity and weak order property for granted, an observed preference relation cannot be described by an SEU model if and only if CCI can be falsified. In normative, or prescriptive, context, it shows that, again taking continuity and weak order for granted, a person can be convinced that he should use the SEU model to determine his behavior, if and only if he can be convinced of the appropriateness of CCI.

Finally some more on strength of preference. We have only considered strength of preference w.r.t. amounts of money. One may also derive a strength of preference relation \succsim^* w.r.t. bets, from \succsim . This is reversed from Sarin (1982), and Theorem 7.4 of Fishburn (1970), where \succsim is derived from \succsim^* by $x \succsim y$ if $(x, y) \succsim^* (y, y)$. Deriving \succsim^* from \succsim is done as follows (where we do not make explicit the assumptions that justify this, they can be found in Sarin, 1982). To see that $(x, y) \succsim^* (v, w)$ we find α, β, γ such that $(y_{-1}\alpha) \approx x$, $(w_{-1}\beta) \approx v$, and $(y_{-2}\gamma) \approx w$. Then $(x, y) \succsim^* (v, w) \Leftrightarrow ((y_{-1}\alpha), y) \succsim^* ((w_{-1}\beta), w) \Leftrightarrow ((y_{-1,2}\alpha, \gamma), (y_{-2}\gamma)) \succsim^* ((w_{-1}\beta), (y_{-2}\gamma)) \Leftrightarrow (y_{-1,2}\alpha, \gamma) \succsim (w_{-1}\beta)$.

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