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# A powerful tool for analyzing concave/convex utility and weighting functions

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**Abstract**

This paper shows that convexity of preference has stronger implications for weighted utility models than had been known hitherto, both for utility and for weighting functions. Our main theorem derives concave utility from convexity of preference on the two-dimensional comonotonic cone, without presupposing continuity. We then show that this, seemingly marginal, result provides the strongest tool presently available for obtaining concave/convex utility or weighting functions. We revisit many classical results in the literature and show that we can generalize and improve them.


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## 1. Introduction

Convexity of preference is a standard condition in many fields (De Giorgi and Mahmoud, 2016; Debreu, 1959; Mas-Colell et al., 1995 p. 44). We examine it for weighted utility models, where its potential has not yet been fully recognized. Our first theorem shows its equivalence to concave utility on the two-dimensional comonotonic cone. This generalizes existing results by not presupposing continuity and by providing flexibility of domain. With this seemingly thin

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and marginal result we can in one blow generalize virtually all existing theorems on convex or concave utility or weighting functions, and make them more appealing.

The aforementioned theorems concern: (1a) risk aversion for expected utility not only for risk (von Neumann and Morgenstern, 1944) but also for (1b) uncertainty (Savage, 1954); (1c) Yaari's (1969) comparative risk aversion generalized by allowing for different beliefs; (2) concave/convex utility and weighting functions (2a) for Gilboa's (1987) and Schmeidler's (1989) rank-dependent utility for ambiguity, and (2b) for Tversky and Kahneman's (1992) prospect theory for ambiguity; (3) corresponding results for Ghirardato and Marinacci's (2001) biseparable utility for ambiguity[1]; (4) smooth ambiguity aversion (Klibanoff et al., 2005). Wakker and Yang (2019) show how our main theorem can be applied to decision under risk,[2] providing results on: (1) concave/convex utility and probability weighting for Quiggin's (1982) rank-dependent utility for risk and Tversky and Kahneman's (1992) prospect theory for risk; (2) corresponding results for Miyamoto's (1988) biseparable utility for risk[3]; (3) loss aversion in Köszegi and Rabin's (2006) reference dependent model; (4) inequality aversion for welfare theory (Ebert, 2004).

The main contribution of this paper is not to generalize some theorems, which would constitute a marginal contribution, but to provide a general technique to obtain convex/concave utility or weighting functions in a more general and appealing manner than done before. As corollaries, we can generalize and improve virtually all existing results on this topic in the literature. To limit the size of this paper, we focus on uncertainty henceforth. Our theorems can readily be applied, though, not only to risk (Wakker and Yang, 2019), but also to discounted utility for intertemporal choice with aversion to variation in outcomes, utilitarian welfare models with aversion to inequality, and other weighted utility models.

The outline of the paper is as follows. Section 2 presents elementary definitions and our main result. To show its usefulness, the following sections apply our main result to a number of well-known classical results in the literature, generalizing and making them more appealing. These applications demonstrate that we have provided a general tool for analyzing concave/convex utility and weighting. Section 3 presents implications for uncertainty focusing on classical expected utility. Sections 4 and 5 turn to ambiguity models, followed by a concluding section and an appendix with proofs. In each proof, we first find a substructure isomorphic to our main theorem, and then extend the desired result to the whole domain considered.

## 2. Definitions and our main theorem

$S$ is the *state space* that can be finite or infinite. $\mathcal{A}$ denotes an algebra of subsets called *events*. The *outcome set* is a nonpoint interval $I \subset I\!R$, bounded or not. $\mathcal{F}$ denotes a set of functions from $S$ to $I$ called *acts*, which are assumed measurable (inverses of intervals are events). Outcomes are

[1] This includes many ambiguity theories, such as maxmin expected utility (Alon and Schmeidler, 2014; Gilboa and Schmeidler, 1989), the $\alpha$-Hurwicz criterion (Arrow and Hurwicz, 1972), multiple priors-multiple weighting (Dean and Ortoleva, 2017), contraction expected utility (Gajdos et al., 2008), alpha-maxmin (Ghirardato et al., 2004; Jaffray, 1994; Luce and Raiffa, 1957 Ch. 13), Hurwicz Expected Utility (Gul and Pesendorfer, 2015), binary RDU (Luce, 2000) including his rank-sign dependent utility, and binary expected utility (Pfanzagl, 1959 pp. 287-288).

[2] We thank an anonymous referee and editor for recommending this organization of our results in two papers.

[3] This is the analog for risk of biseparable utility, and includes many risk theories such as disappointment theory (Bell, 1985; Loomes and Sugden, 1986) for a disappointment function kinked at 0, RAM and TAX models (Birnbaum, 2008), disappointment aversion (Gul, 1992), original prospect theory (Kahneman and Tversky, 1979) for gains and for losses, Luce's (2000) binary RDU, and prospective reference theory (Viscusi, 1989). See Wakker (2010 Observation 7.11.1).

identified with constant acts. We assume that $\mathcal{F}$ contains all *simple*, i.e., finite-valued measurable functions. Other than that, $\mathcal{F}$ can be general, with however the restriction added that all RDU values (defined later) are well-defined. In particular, $\mathcal{F}$ may consist exclusively of simple acts, or contain all bounded acts. By $x = (E_1 : x_1, \dots, E_n : x_n)$ we denote the simple act assigning outcome $x_j$ to all states in $E_j$. It is implicitly understood that the $E_j$s are events partitioning $S$.

A *preference relation*, i.e., a binary relation $\succcurlyeq$ on $\mathcal{F}$, is given; $\succ, \preccurlyeq, \prec, \sim$ are as usual. $V$ *represents* $\succcurlyeq$ on $\mathcal{F}' \subset \mathcal{F}$ if $V$ is real valued with $\mathcal{F}'$ contained in its domain and $x \succcurlyeq y \Leftrightarrow V(x) \geq V(y)$ for all acts $x, y \in \mathcal{F}'$. This implies weak ordering on $\mathcal{F}'$, i.e., $\succcurlyeq$ is transitive and complete there. If we omit "on $\mathcal{F}'$", then $\mathcal{F}' = \mathcal{F}$. Central in this paper are convex combinations $\lambda x + (1-\lambda)y$. Here $x$ and $y$ are acts, $0 \leq \lambda \leq 1$, and the combination concerns the statewise mixing of outcomes. We do not assume that $\mathcal{F}$ is closed under convex combinations. The set of simple acts is, and this provides enough richness for all our theorems.

**Definition 1.** We call $\succcurlyeq$ *convex* if $x \succcurlyeq y \Rightarrow \lambda x + (1-\lambda)y \succcurlyeq y$ for all $0 \leq \lambda \leq 1$ and acts $x, y, \lambda x + (1-\lambda)y$. □

The condition is only imposed if the mix indeed is an act; i.e., is contained in the domain. Convexity of preference is equivalent to quasi-concavity of representing functions.[4]

An *(event) weighting function* $W$ maps events to $[0, 1]$ such that: $W(\emptyset) = 0$, $W(S) = 1$, and $A \supset B \Rightarrow W(A) \geq W(B)$. *Finitely additive probability measures* $P$ are additive weighting functions. They need not be countably additive. Preference conditions necessary and sufficient for countable additivity are well known (Arrow, 1971; Wakker, 1993 Proposition 4.4), and can optionally be added in all our theorems.

For a weighting function $W$ and a function $U : I \to \mathbb{R}$, the *rank-dependent utility* (*RDU*) of an act $x$ is

$$\int_{\mathbb{R}^+} W\{s \in S : U(x(s)) > \alpha\} d\alpha - \int_{\mathbb{R}^-} (1 - W\{s \in S : U(x(s)) > \alpha\}) d\alpha. \tag{1}$$

An alternative term used in the literature is Choquet expected utility. We impose one more restriction on $\mathcal{F}$: RDU is well defined and finite for all its elements. A necessary and sufficient condition directly in terms of preferences—requiring preference continuity with respect to truncations of acts—is in Wakker (1993). A sufficient condition is that all acts are bounded (with an upper and lower bound contained in $I$). For a simple act $(E_1 : x_1, \dots, E_n : x_n)$ with $x_1 \geq \dots \geq x_n$, the RDU is

$$\sum_{j=1}^{n} (W(E_1 \cup \dots \cup E_j) - W(E_1 \cup \dots \cup E_{j-1}))U(x_j). \tag{2}$$

*Rank-dependent utility* (*RDU*) holds on $\mathcal{F}' \subset \mathcal{F}$ if there exist $W$ and strictly increasing $U$ such that $RDU$ represents $\succcurlyeq$ on $\mathcal{F}'$. Then $U$ is called the *utility function*. Again, if we omit "on $\mathcal{F}'$," then $\mathcal{F}' = \mathcal{F}$. If $\mathcal{F}'$ contains all constant acts, then strict increasingness of $U$ is equivalent to *monotonicity*: $\gamma > \beta \Rightarrow \gamma \succ \beta$ for all outcomes. We do not require continuity of $U$. The special case of RDU with $W$ a finitely additive probability measure $P$ is called *expected utility* (*EU*). We sometimes write *subjective EU* if $P$ is subjective. We assume existence of a *nondegenerate*

[4] Unfortunately, terminology in the literature is not uniform, and sometimes terms concave, quasi-convex, or quasi-concave have been used. We use the most common term, convex.

*event* $E$, meaning $(E:\gamma, E^c:\gamma) \succ (E:\gamma, E^c:\beta) \succ (E:\beta, E^c:\beta)$ for some outcomes $\gamma > \beta$ with the acts contained in the relevant domain $\mathcal{F}'$. Under sufficient richness, satisfied in all cases considered in this paper, nondegenerateness means $0 < W(E) < 1$.[5] We summarize the assumptions made.

**Assumption 2.** [Structural assumption for uncertainty] $S$ is a state space, $\mathcal{A}$ an algebra of subsets (events), and $I$ a nonpoint interval. $\mathcal{F}$, the set of acts, is a set of measurable functions from $S$ to $I$ containing all simple functions, endowed with a binary (preference) relation $\succcurlyeq$. RDU represents $\succcurlyeq$ on a subset $\mathcal{F}'$ of $\mathcal{F}$ (default: $\mathcal{F}' = \mathcal{F}$). There exists a nondegenerate event $E$. □

To obtain complete preference axiomatizations in the theorems in this paper, we should state preference conditions for the decision models assumed. Such conditions were surveyed by Köbberling and Wakker (2003) and will not be repeated here.

$W$ is *convex* if $W(A \cup B) + W(A \cap B) \geq W(A) + W(B)$. Elementary manipulations show that this holds if and only if

$$W(A \cup B) - W(B) \leq W(A \cup B') - W(B') \text{ whenever } A \cap B' = \emptyset \text{ and } B \subset B'. \tag{3}$$

The latter formulation shows the analogy with increasing derivatives of real-valued convex functions. An interesting implication of convexity of $W$ is that RDU then belongs to the popular maxmin EU model (Wald, 1950; Gilboa and Schmeidler, 1989) with the set of priors equal to the *Core*, i.e., the set of probability measures that dominate $W$ (Schmeidler, 1986 Proposition 3; Shapley, 1971).

The following theorem is our main result. Virtually all preceding results in the literature—Debreu and Koopmans (1982) excepted—assumed continuity and often even differentiability, but we do not.

**Theorem 3.** *[Main theorem] Assume: (a) Structural Assumption 2; (b) $S = \{s_1, s_2\}$; (c) $s_1$ is nondegenerate; (d) EU (= RDU) holds on $\mathcal{F}' = \{x = (s_1 : x_1, s_2 : x_2) : x_1 \geq x_2\}$. Then utility is concave if and only if $\succcurlyeq$ is convex on $\mathcal{F}'$.* □

In the theorem, nondegeneracy of $s_1$ is equivalent to nondegeneracy of $s_2$. The proof of the theorem is more complex than of its analogs on full product spaces that can use hedging, as in the half-half mixture of $(1, 0)$ and $(0, 1)$ resulting in the sure $(0.5, 0.5)$. Hedging provides a powerful tool for analyzing convex preferences, extensively used in the literature, that we cannot use though because all acts in our domain are maximally correlated. This complicates our proof relative to, for instance, Debreu and Koopmans (1982), its simplification Crouzeix and Lindberg (1986), its generalization Monteiro (1999), and most other predecessors. Therefore, unlike Debreu and Koopmans, we need strictly increasing utility. Example A.3 shows that our theorem does not hold for nondecreasing utility. Because strictly increasing utility is natural in most applications, it does not entail a serious restriction. In return, the flexibility of domain provided by

[5] If no nondegenerate event exists, then an RDU representation exists with a linear, so surely concave, utility, and in this sense all results below hold true—also regarding convexity of weighting functions as can be demonstrated—without the extra requirement of nondegeneracy. However, then utility is ordinal (Wakker, 1989 Observation VI.5.1') so that utility can also be chosen nonconcave, and we should formulate all our results as existence results. We avoid complicating our formulations this way.

our theorem allows us to apply it to utility functions when expected utility is violated, and to apply it dually so that it speaks to weighting functions. Chateauneuf and Tallon (2002), Ghirardato and Marinacci (2001), and Wakker (1994) did consider comonotonic sets of acts (defined in §4) as above. Theorem 3 generalizes their results by showing that continuity/differentiability is redundant. §5 gives further details.

## 3. Implications for decision under uncertainty: expected utility

This section considers applications of the main Theorem 3 to classical EU for decision under uncertainty.

**Corollary 4.** *If Structural Assumption 2 and EU hold, then $U$ is concave if and only if $\succcurlyeq$ is convex.* □

Corollary 4 is useful for capturing risk aversion because convexity is directly observable, not involving subjective probabilities. Remarkably, the early Yaari (1965) already pointed out that the traditional definitions of risk aversion, relating to expected value or mean-preserving spreads, cannot be used for subjective EU. He hence tested convexity instead. However, he did not observe that convexity is actually equivalent to the traditional definitions.

Although an early version of Corollary 4 appeared in Debreu and Koopmans (1982 p. 4) and has been used in some works (Section 5), the result did not yet receive the attention it deserves and has not been generally known. Alternative, more complex, preference conditions for concave utility under subjective EU are in Baillon et al. (2012), Harvey (1986 Theorem 3), Wakker (1989), Wakker (2010 Eq. 4.8.2), and Wakker and Tversky (1993 §9).

We next turn to comparative results. In what follows, superscripts refer to decision makers. Yaari (1969) provided a well-known characterization of comparative risk aversion under subjective EU, where decision maker $\succcurlyeq^2$ with utility $U^2$ is more risk averse than decision maker $\succcurlyeq^1$ with utility $U^1$ if her certainty equivalents are always lower. Then $U^2$ is a concave transformation of $U^1$. Unfortunately, Yaari's condition is not necessary and sufficient, but only holds if the two decision makers have the same subjective probabilities. Decision makers with different beliefs cannot be compared because Yaari's condition then is never satisfied. The basic problem is that certainty equivalents depend on probabilities and, thus, involve not only risk attitudes but also beliefs. Yaari's method of comparing certainty equivalents has become a common tool in ambiguity theories, for instance to compare ambiguity aversion across decision makers.[6] Then invariably all other attitude components of the decision makers except the one compared have to be identical. This is implied by the fundamental problem of certainty equivalents of involving all components of decision attitudes. It limits the scope of application. We now show how outcome mixing avoids the aforementioned limitations and works for general beliefs, for Yaari's original EU framework. Generalizations to nonexpected utility theories are left to future work. A preparatory notation: $\alpha_E\beta$ denotes the binary act $(E:\alpha, E^c:\beta)$. Mathematically, we will describe the case where $\succcurlyeq^2$ is risk averse if outcomes are expressed in $U^1$ units, i.e., units that make $\succcurlyeq^1$ risk neutral. In these outcome units, $\succcurlyeq^2$ should be convex. To capture this idea in a preference condition, we have to avoid the explicit use of theoretical constructs such as $U^1$. We

[6] See, for instance, Epstein (1999 Definition 2.3), Ghirardato and Marinacci (2001 §4.1), Ghirardato and Marinacci (2002 Definition 4), Izhakian (2017 Definition 4), and Klibanoff et al. (2005 Definition 5).

have to reveal mixtures of acts in $U^1$ units directly from preferences. Gul (1992) showed a way to do this. Assume, for any event $A$, with $x_j \geq y_j \geq z_j$ for $j = 1, 2$:

$$(x_{1_A} z_1) \sim^1 y_1 \textit{ and } (x_{2_A} z_2) \sim^1 y_2. \tag{4}$$

This shows that, in $U^1$ units, $y_1$ is a mixture of $x_1$ and $z_1$, and $y_2$ is so of $x_2$ and $z_2$, with the subjective probabilities $P^1(A)$ and $1 - P^1(A)$ as mixing weights. These weights are not directly observable but this is no problem. All we need for what follows is that these weights are the same in both mixtures. This is enough to infer that, for any event $B$, the act $(y_{1_B} y_2)$ is a convex mixture of $(x_{1_B} x_2)$ and $(z_{1_B} z_2)$ in $U^1$ units. Our convexity condition requires that the mixture $(y_{1_B} y_2)$ is preferred to the other two acts if they are indifferent. That is, $\succcurlyeq^2$ is *more outcome-risk averse*[7] than $\succcurlyeq^1$ if, for all events $B$:

$$(x_{1_B} x_2) \sim^2 (z_{1_B} z_2) \Rightarrow (y_{1_B} y_2) \succcurlyeq^2 (z_{1_B} z_2) \tag{5}$$

is implied by Eq. (4). Eq. (5) is the convexity condition in terms of $U^1$ units, weakened to the case where the antecedent preference is actually an indifference, and where the mixture weights are $P^1(A)$ and $1 - P^1(A)$ for some event $A$. Under continuity, this weakened version of convexity is strong enough to imply full-force convexity, as Corollary 5 will show. To see intuitively that our condition captures comparative risk aversion, first note that Eq. (4) implies, for all events $B'$

$$(x_{1_{B'}} x_2) \sim^1 (z_{1_{B'}} z_2) \Rightarrow (y_{1_{B'}} y_2) \sim^1 (z_{1_{B'}} z_2) \tag{6}$$

because of linearity in probability mixing of the $EU^1$ functional.[8] The event $B'$ to bring indifference for decision maker 2 may be different than $B$ due to different beliefs and/or state spaces. Comparing Eqs. (5) and (6) shows that, if decision maker 1 is indifferent to the mix, then decision maker 2 prefers it. This reveals stronger risk aversion, as formalized next.

**Corollary 5.** *Assume that $\succcurlyeq^1$ and $\succcurlyeq^2$ both satisfy Structural Assumption 2 with the same outcome interval $I$, and both maximize subjective EU with continuous utility functions $U^1$ and $U^2$, respectively. Then $\succcurlyeq^2$ is more outcome-risk averse than $\succcurlyeq^1$ if and only if $U^2(.) = \varphi(U^1(.))$ for a concave transformation $\varphi$. The two decision makers may have different probabilities and may even face different state spaces.* □

Both the ambiguity aversion of Klibanoff et al.'s (2005) smooth model, and the preference for early resolution of uncertainty of Kreps and Porteus (1978) amount to having one EU utility function more concave than another. Corollary 5 shows a way to obtain these results without involving probabilities as inputs in the preference condition. Note that we have to be able to deal with different state spaces in these applications.

Our outcome-risk aversion condition is more complex, and less appealing, than Yaari's certainty equivalent condition. Its pro is that it delivers a clean comparison of utility and risk attitude, not confounded by beliefs. Both conditions deserve study. Baillon et al. (2012) provided other characterizations of comparative risk aversion that, like our result, do not require same beliefs or given probabilities. They used an endogenous midpoint operation for utilities. Heufer (2014) showed how Yaari's certainty equivalent condition can be elicited from revealed preferences. Our paper propagates the use of preference convexity. Heufer (2012) showed how this convexity can be elicited from revealed preferences.

[7] To avoid confusion with Yaari's widely accepted terminology, we add "outcome" to our term.

[8] Gul (1992) used a strengthened version of the implication Eq. (4) ⇒ Eq. (6) to axiomatize subjective expected utility.

## 4. Implications for decision under uncertainty: ambiguity

We first discuss applications to RDU, and start with basic definitions. Acts $x, y$ are *comonotonic* if there are no states $s, t$ with $x(s) > x(t)$ and $y(s) < y(t)$. A set of acts is *comonotonic* if each pair of its elements is comonotonic. A *comoncone* is a maximal comonotonic set. It corresponds with an ordering $\rho$, called *ranking*, of $S$ and contains all acts $x$ with $s\rho t \Rightarrow x(s) \geq x(t)$. Every event $E$ of the form $\{s \in E : s\rho t, s \neq t\}$ is called a *goodnews* event or, more formally, a *rank*. Intuitively, it reflects the good news of receiving all outcomes ranked better than some outcome. A set of acts is comonotonic if and only if it is a subset of a comoncone. A comoncone is *nondegenerate* if it has a nondegenerate goodnews event. On every comoncone with related ordering $\rho$, RDU agrees with an EU functional with the finitely additive probability measure $P_\rho$ agreeing with $W$ on the goodnews events. The following corollary is a straightforward generalization of the main Theorem 3.

**Corollary 6.** *If Structural Assumption 2 holds and RDU holds on a nondegenerate comoncone $\mathcal{F}'$, then $U$ is concave if and only if $\succcurlyeq$ is convex on $\mathcal{F}'$.* $\square$

A first version of the following result was in Chateauneuf and Tallon (2002), and did not receive the attention it deserves.

**Corollary 7.** *[Main corollary] If Structural Assumption 2 and RDU hold, then $\{U$ is concave and $W$ is convex$\}$ if and only if $\succcurlyeq$ is convex.* $\square$

The result is especially appealing because the two most important properties of RDU jointly follow from one very standard preference condition. The only proof available as yet, by Chateauneuf and Tallon, assumes differentiability of utility, which is problematic for preference foundations. Differentiability is not problematic, and useful, in most economic applications. For preference foundations the situation is different, though. Preference foundations seek for conditions directly observable from preferences. In this way, preference foundations make theories operational. For general differentiability, there is no clear and elementary preference condition.[9] Hence, differentiability assumptions are better avoided in preference foundations. Strictly speaking, Corollary 7 is then the first joint preference foundation of the two most popular specifications of RDU. Chateauneuf and Tallon (2002) did not present their result very saliently.[10] This, and the use of differentiability, may explain why this appealing result has not yet been as widely known as it deserves to be.

We, finally, give results for a large class of nonexpected utility models. By $\mathcal{F}_E$ we denote the set of binary acts $\gamma_E\beta = (E : \gamma, E^c : \beta)$, and by $\mathcal{F}_E^{\uparrow}$ we denote the subset with $\gamma \geq \beta$. *Biseparable utility* holds if there exist a utility function $U$, and a weighting function $W$, such that $RDU(x)$ represents $\succcurlyeq$ on the set of all binary acts $x$. That is, for all binary acts $\gamma_E\beta$ $(\gamma \geq \beta)$ we have an RDU representation $W(E)U(\gamma) + (1 - W(E))U(\beta)$, but for acts with more than two outcomes the representation has not been restricted. Biseparable utility includes many theories (see the

9 Once concavity has been derived, we are close to differentiability (lemma A.1). Then necessary and sufficient conditions for complete differentiability can be stated in terms of vanishing limits of risk premia (Nielsen, 1999), a condition which has the same, commonly accepted, observability status as continuity. The task of our paper is, however, to derive concavity.

10 One has to combine their Proposition 1 with the equivalence of (i) and (iv) in their Theorem 1.

Introduction) and the following theorem therefore pertains to all these theories. Statement (ii) characterizes concave utility for all these models. Statement (i) additionally characterizes *subadditivity* of $W$: $W(A) + W(A^c) \leq 1$. It is equivalent to convexity on binary partitions $\{E, E^c\}$. It is the prevailing empirical finding and has sometimes been taken as definition of ambiguity aversion.

**Corollary 8.** *If Structural Assumption 2 and biseparable utility hold, then:*

(i) *$U$ is concave and $W$ is subadditive if and only if $\succcurlyeq$ is convex on every $\mathcal{F}_E$.*
(ii) *$U$ is concave if and only if $\succcurlyeq$ is convex on every $\mathcal{F}_E^{\uparrow}$. This holds if and only if $\succcurlyeq$ is convex on one set $\mathcal{F}_E^{\uparrow}$ with $E$ nondegenerate.* □

## 5. Further implications for existing results on uncertainty in the literature

Schmeidler (1989), the most famous work in ambiguity theory, assumed an Anscombe-Aumann framework: a set of prizes is given, and the outcome set is the set of simple probability distributions over the prize space. That is, the outcome set is a convex subset of a linear space. Acts map states to outcomes. Utility over outcomes is assumed to be expected utility, i.e., it is linear with respect to probabilistic mixtures of outcomes. This is Structural Assumption 2 but with a multi-dimensional outcome space instead of our one-dimensional $I$. Schmeidler (1989 the Proposition) showed that $W$ is convex if and only if $\succcurlyeq$ is convex—called uncertainty aversion. This follows from the special case of the main Corollary 7 with utility linear. That the outcome space is multi-dimensional changes nothing in our proofs.[11]

Most studies of multiple priors models, including $\alpha$−maxmin models, used the Anscombe-Aumann framework with linear utility. Exceptions without this restriction include Alon and Schmeidler (2014), Casadesus-Masanell et al. (2000), and Ghirardato et al. (2003). Our results characterize concave utility for the latter studies.

Several studies assumed linear utility as did Schmeidler (1989), and then gave various necessary and sufficient conditions for convex weighting functions alternative to our convexity: Chateauneuf (1991) and Kast and Lapied (2003) for monetary outcomes, and Wakker (1990) for the Anscombe-Aumann framework.

Three results in the literature come close to our main Theorem 3, in deriving concavity not on a full product set but on a comoncone. The first is Wakker (1994 Theorem 24).[12] He applied our main theorem dually to probability weighting instead of utility (similarly as our derivation of convexity of $W$ in Corollary 7). His proof was complex and heavily used continuity in probability weighting and utility. His result will be generalized by Wakker and Yang (2019), who also formalize the aforementioned duality. The second result close to our main Theorem 3 or, more precisely, to our Corollary 6, is Theorem 3 in Chateauneuf and Tallon (2002). They assumed differentiable utility, whence they could skip Steps 2 and 3 of our proof in Appendix A. The third result close to Theorem 3 is Ghirardato and Marinacci (2001 Theorem 17). They assumed

[11] For necessity of the preference condition, the proof of Lemma B.1 works with the first inequality an equality. For sufficiency, all $o$ terms in the proof of Lemma B.2 are exactly 0. Sufficiency can also be obtained by taking two outcomes $\gamma \succ \beta$ and equating lotteries over them with $I = [0, 1]$.

[12] Wakker (1994) used the term quasi-concave preference instead of our term convex preference.

continuity, and showed how Debreu and Koopmans (1982) can be used as in Step 1 of our proof in Appendix A. Our proof shows how to add Steps 2 and 3 to their proof.[13]

For the special case of linear utility (in an Anscombe-Aumann framework), Cerreia-Vioglio et al. (2011) characterized general preference functionals with convex preferences. Rigotti et al. (2008) examined general convex preferences and specified results for several ambiguity models. Their Remark 1 discussed RDU with convex weighting functions, but did not specify how these are related to convex preferences. Our main Corollary 7 shows that concavity of utility is necessary and sufficient for that relation to be an equivalence.

## 6. Conclusion

We have provided a general technique to obtain convex/concave utility and weighting functions. Fields of application include intertemporal choice, utilitarian welfare aggregations, risk, and, the context chosen in this paper, decision under uncertainty. There this paper generalized and improved virtually all existing theorems, and Wakker and Yang (2019) will do so for risk. Knowledge of Corollaries 4 and 7 will be useful for everyone working in decision theory. Convexity with respect to outcome-mixing is more powerful than had been known before.

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## Appendix A. Proof of Theorem 3

We first list some well-known properties of concave functions (Van Rooij and Schikhof 1982 §1.2).

**Lemma A.1.** *If $U$ is concave and strictly increasing on $I$, then: (a) $U$ is continuous on $I$ except possibly at $min(I)$ (if it exists). On int(I): (b) $U$ has right derivative $U'_r$ and left derivative $U'_\ell$ everywhere; (c) $U'_\ell(\alpha) \geq U'_r(\alpha) \geq U'_\ell(\alpha') > 0$ for all $\alpha' > \alpha$; (d) $U$ is differentiable almost everywhere.*

**Proof.** As regards positivity in (c), if a left or right derivative were 0 somewhere in $int(U)$ then it would be 0 always after, contradicting strict increasingness of $U$. The other results are well-known (Van Rooij and Schikhof, 1982 §1.2). □

**Proof of main Theorem 3.** If $U$ is concave then so is the EU functional, so that it is quasi-concave, implying convexity of $\succcurlyeq$. (This also follows from Lemma B.1.)

In the rest of this appendix we assume convexity of $\succcurlyeq$, and derive concavity of $U$. We write $\pi_1 = W(s_1), \pi_2 = 1 - \pi_1$. By nondegeneracy, $0 < \pi_1 < 1$. We suppress states from acts and write $(x_1, x_2)$ for $(s_1 : x_1, s_2 : x_2)$. As explained in the main text, we cannot use hedging techniques in our proofs. Instead, we will often derive contradictions of convexity of $p$ by constructing a "risky" act $r = (r_1, r_2)$ and a "close-to-certain" act $c = (c_1, c_2)$ with $r_1 > c_1 \geq c_2 > r_2$, such that

[13] We thus show that the claim $X^0 \subset X^* \cup X_*$ on p. 887 line $-13$ in their proof holds true by ruling out the existence of $\beta$ as in Figs. 2 and 3.

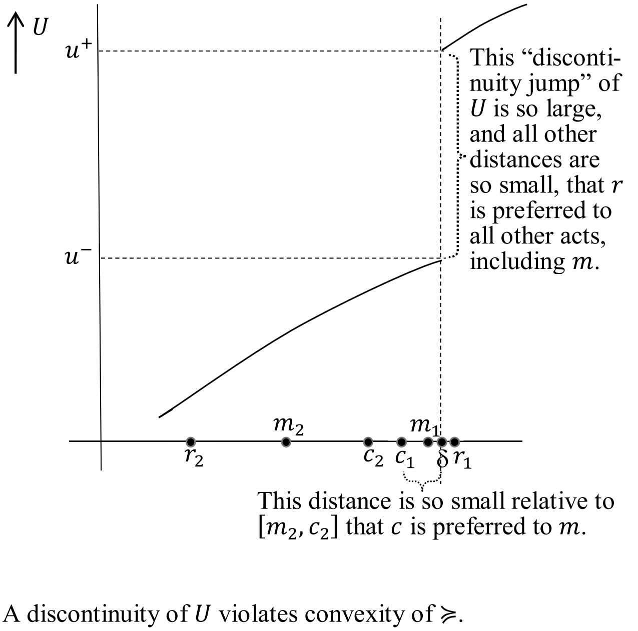



Fig. 1. Proof of continuity of $U$.

$$m := \lambda r + (1-\lambda)c;\ \ r \succ m; c \succ m \tag{A.1}$$

for some $0 < \lambda < 1$ (mostly $\lambda = 0.5$), with $m$ called the "middle" act.

**Lemma A.2.** *$U$ is continuous except possibly at $inf\ I$.*

**Proof.** See Fig. 1. Assume, for contradiction, that $U$ is not continuous at an outcome $\delta > inf\ I$. We construct $r, m, c$ as in Eq. (A.1) with further $r_1 \geq \delta > m_1$.

Define $u^- = sup\{U(\beta) : \beta < \delta\}$. Define $u^+ = U(\delta)$ if $\delta = max(I)$ and $u^+ = inf\{U(\alpha) : \alpha > \delta\}$ otherwise. By discontinuity, $u^+ > u^-$. By taking $r_2 < \delta$ sufficiently close to $\delta$ we can get $U(r_2)$ as close to $u^-$ as we want. We take it so close that $\pi_1(u^+) + \pi_2 U(r_2) > u^-$. This will ensure that $r$, the only act with its first outcome exceeding $\delta$, is strictly preferred to all other acts, in particular, to $m$. We next choose $c_2$ strictly between $r_2$ and $\delta$ and define $m_2 = (r_2 + c_2)/2$. We then take $c_1$ strictly between $c_2$ and $\delta$ so close to $\delta$ that $\pi_1(u^-) + \pi_2 U(m_2) < \pi_1 U(c_1)) + \pi_2(U(c_2))$. This will ensure that $c \succ m$ if we ensure that $m_1 < \delta$. For the latter purpose we define $r_1 = \delta$ if $\delta = max(I)$, and otherwise $r_1 > \delta$ so close to $\delta$ that $m_1 := (r_1 + c_1)/2 < \delta$. In both cases, $U(r_1) \geq u^+$. The acts $r, m, c$ are as in Eq. (A.1) with $\lambda = 1/2$. *QED*

Because $U$ is strictly increasing, it suffices to prove concavity outside $inf\ I$. That is, we assume that $I$ has no minimum. Assume for contradiction that $U$ is not concave. Then there exist $0 < \lambda' < 1$ and outcomes $\alpha' < \gamma'$ such that $U(\lambda'\gamma' + (1-\lambda')\alpha') < \lambda' U(\gamma') + (1-\lambda')U(\alpha')$. Define $\ell$ as the line through $(\alpha', U(\alpha'))$ and $(\gamma', U(\gamma'))$. By continuity of $U$, we can define $\alpha$ as the maximum outcome between $\alpha'$ and $\lambda'\gamma' + (1-\lambda')\alpha'$ with $(\alpha, U(\alpha))$ on (or above) $\ell$, and $\gamma$

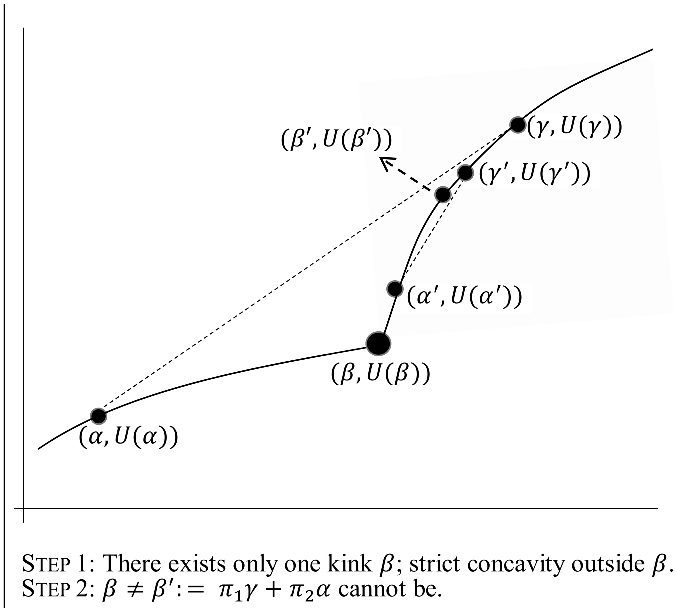



Fig. 2. Contradiction from nonconcavity of $U$: Steps 1 & 2.

as the minimum outcome between $\gamma'$ and $\lambda'\gamma' + (1-\lambda')\alpha'$ with $(\gamma, U(\gamma))$ on (or above) $\ell$. We have $\gamma > \alpha$ and

$$U(\lambda\gamma + (1-\lambda)\alpha) < \lambda U(\gamma) + (1-\lambda)U(\alpha) \tag{A.2}$$

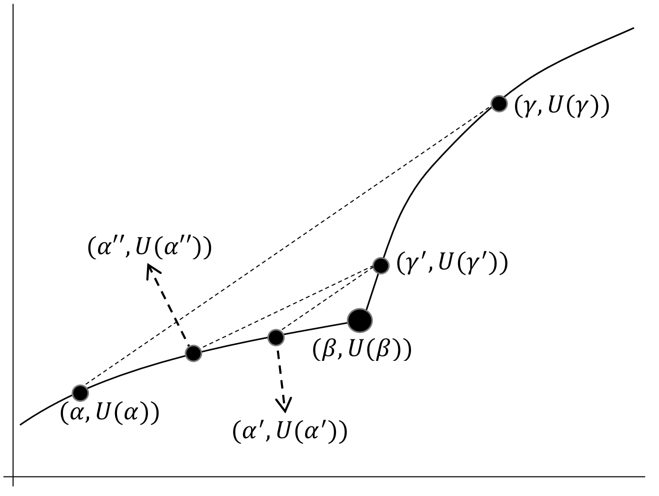
for all $0 < \lambda < 1$ (Fig. 2).

STEP 1 [At most one nonconcavity kink]. Assume that $U$ is not concave on some "middle" interval $M \subset I$, with $L$ and $R$ the intervals in $I$ to the left and right of $M$, possibly empty. Applying Debreu and Koopmans (1982 Theorem 2) to the additive representation $(1-\pi_1)U(x_1)+\pi_1 U(x_2)$ on $L \times M$ and to the additive representation $(1-\pi_1)U(x_1)+\pi_1 U(x_2)$ on $M \times R$ then implies that $U$ is strictly concave on $L$ and $R$. Applying this result to smaller and smaller subintervals $M$ of $[\alpha, \gamma]$ there is one $\beta$ in $[\alpha, \gamma]$ such that $U$ is concave above and below $\beta$.

For the remainder of the proof, we could use Debreu and Koopmans (1982 Theorem 6). They provide a concavity index according to which $U$ would be infinitely *convex* at the nonconcavity kink, then would have to be more, so infinitely, concave at every other point, but being concave there it can be infinitely concave at no more than countably many points, and a contradiction has resulted. This reasoning is advanced and cannot be written formally very easily, because of which we provide an independent proof.

STEP 2 [Nonconcavity kink must be exactly at $\pi_1\gamma + \pi_2\alpha$]. We next show that $\beta$ can only be $\beta' := \pi_1\gamma + \pi_2\alpha$. Assume, for contradiction, that $\pi_1\gamma + \pi_2\alpha$ were located at a point $\beta'$ different than $\beta$ (Fig. 2). Then there would be a small interval around $\beta'$, not containing $\alpha$, $\beta$, or $\gamma$, where $U$ would be strictly concave. We could then find $\gamma' > \beta' > \alpha'$ in the small interval with exactly $\pi_1\gamma' + \pi_2\alpha' = \beta'$ and satisfying the strict concavity inequality

$$\pi_1 U(\gamma') + \pi_2 U(\alpha') < U(\beta'). \tag{A.3}$$




STEP 3: We replace $\alpha$ by $\alpha''$ and then Step 2 is violated.

Fig. 3. Contradiction from nonconcavity of $U$: Step 3.

We define some acts, where, on our domain $\mathcal{F}'$ considered, always the best outcome is the first: $c = (\beta', \beta')$, $r = (\gamma, \alpha)$, $m = (\gamma', \alpha')$, and we show that Eq. (A.1) holds. Because $\beta'$ is the same $\pi_1, \pi_2$ mix of $\gamma$ and $\alpha$ as of $\gamma'$ and $\alpha'$, there exists $0 < \mu < 1$ such that $m = \mu r + (1-\mu)c$. We have $r \succ c \succ m$, the first preference by the convexity "risk-seeking" inequality Eq. (A.2) (with $\lambda = \pi_1$) and the second by the concavity "risk-averse" inequality Eq. (A.3). Eq. (A.1) is satisfied.

STEP 3 [Nonconcavity kink cannot be at $\pi_1\gamma + \pi_2\alpha$ either]. We construct, for contradiction, two outcomes $\gamma', \alpha'$ or $\gamma', \alpha''$ that are as $\gamma, \alpha$ in Eq. (A.2) but for whom an analogous $\beta$ as just constructed does not exist though. See Fig. 3.

Consider the line through $(\alpha, U(\alpha))$ and $(\gamma, U(\gamma))$. Because of concavity of $U$ on $[\alpha, \beta)$ and $[\beta, \gamma)$, the right derivative of $U$ is decreasing on each of these two intervals. Because the line mentioned lies above the graph of $U$, its slope exceeds the right derivative of $U$ everywhere on $[\alpha, \beta)$ and is below it everywhere on $[\beta, \gamma)$. Now take any $\alpha'$ strictly between $\alpha$ and $\beta$, and take the line parallel to the line through $(\alpha, U(\alpha))$ and $(\gamma, U(\gamma))$. Its first intersection with the graph of $U$ exceeding $\alpha'$ is $\gamma'$. Inspection of right derivates shows that $\gamma' > \beta$. If $\pi_1\gamma' + \pi_2\alpha' \neq \beta$ then we have a contradiction with Step 2 and we are done. If we have equality after all, then we take a line through $(\gamma', U(\gamma'))$ that is strictly between the two parallel lines. Its first intersection with the graph of $U$ is $(\alpha'', U(\alpha''))$, and $\alpha''$ and $\gamma'$ are as in Eq. (A.2) but the outcome $\beta$ has $\pi_1\gamma' + \pi_2\alpha'' \neq \beta$. Contradiction has resulted and we are done. The proof of Theorem 3 now is complete. □

The following example shows that the main Theorem 3 does not hold if $U$ is only assumed nondecreasing instead of strictly increasing.

**Example A.3.** In this example, all assumptions of Theorem 3 are satisfied except that $U$ is nondecreasing instead of strictly increasing. $\succcurlyeq$ is convex but $U$ is not concave.

$I = I\!R$, $W(s_1) = \pi_1 = \pi_2 = 0.5$, and $U$ is the sign function. More precisely, $U(\alpha) = 1$ if $\alpha > 0$, $U(0) = 0$, and $U(\alpha) = -1$ if $\alpha < 0$. We show that $\succcurlyeq$ is convex. We consider an exhaustive list of cases of three acts $m, r, c$, with the cases ordered by the preference value of $m$. In each case we state some implications from $m \prec r$ and $m \prec c$ that readily preclude $m$ from being a mixture of $r$ and $c$. We suppress states and denote acts by the utilities of their outcomes. Thus, $(1, -1)$ denotes an act with a positive first outcome and a negative second outcome. We write $d'_j = U(d_j)$ for $d = c, m, r$ and $j = 1, 2$.

(1) If $m'_1 = -1$ then $r'_1 \geq 0$ and $c'_1 \geq 0$. (2) If $(m'_1, m'_2) = (0, -1)$ then $r'_1 \geq 0$ and $c'_1 \geq 0$ and either one of the latter is positive or $(r'_1, r'_2) = (c'_1, c'_2) = (0, 0)$. (3) If $(m'_1, m'_2) = (0, 0)$ or $(m'_1, m'_2) = (1, -1)$ then both $(r'_1, r'_2)$ and $(c'_1, c'_2)$ are $(1, 0)$ or $(1, 1)$. (4) If $(m'_1, m'_2) = (1, 0)$ then $(r'_1, r'_2) = (c'_1, c'_2) = (1, 1)$. (5) $(m'_1, m'_2) = (1, 1)$ cannot be. $\square$

## Appendix B. Proofs for §3 and §4

For a weighting function $W$, we use the following notation: $\pi(E^R) = W(E \cup R) - W(R)$ is the *decision weight* of an outcome in RDU under event $E$ if $R$ is the *rank*, i.e., the event giving outcomes ranked better than the one under $E$. $\pi^b(E) = W(E)$ and $\pi^w(E) = 1 - W(E^c)$. $W$ is additive (EU) if and only if we have rank independence, i.e., decision weights $\pi(E^R)$ do not dependend on the rank $R$. $W$ is convex if and only if decision weights $\pi(E^R)$ are nondecreasing in the rank $R$ (Eq. (3)), which reflects a pessimistic attitude. If $W$ is convex, then its Core consists of all probability measures $P_\rho$ (defined in §4) and

$$\text{RDU is the infimum EU with respect to all } P_\rho. \tag{B.1}$$

The following observation provides sufficiency of the preference conditions in all our results.

**Lemma B.1.** *If $U$ is concave and $W$ is convex, then $RDU$ is concave and, hence, $\succcurlyeq$ is convex on $\mathcal{F}$ (thus on every $\mathcal{F}' \subset \mathcal{F}$).*

**Proof.** Consider three acts $x, y, \lambda x + (1 - \lambda)y$, contained in comoncones with orderings $\rho_x, \rho_y, \rho_\lambda$, respectively. Then, with the first inequality due to concave utility and the second due to convexity of $W$ (Eq. (B.1)): $RDU(\lambda x + (1 - \lambda)y) = \int_S (U(\lambda x + (1 - \lambda)y))dP_{\rho_\lambda} \geq \int_S (\lambda U(x) + (1 - \lambda)U(y))dP_{\rho_\lambda} = \lambda \int_S U(x)dP_{\rho_\lambda} + (1 - \lambda)\int_S U(y)dP_{\rho_\lambda} = \lambda EU_{P_{\rho_\lambda}}(x) + (1 - \lambda)EU_{P_{\rho_\lambda}}(y) \geq \lambda EU_{P_{\rho_x}}(x) + (1 - \lambda)EU_{P_{\rho_y}}(y) = \lambda RDU(x) + (1 - \lambda)RDU(y)$. $\square$

**Proof of Corollary 4.** Necessity of the preference condition follows from Lemma B.1. Sufficiency follows from applying the main Theorem 3 to any two-dimensional subspace $\{\alpha_E\beta \in \mathcal{F} : \alpha \geq \beta\}$ with $E$ nondegenerate. $\square$

**Proof of Corollary 5.** Both utilities being strictly increasing and continuous, we define the continuous strictly increasing $\varphi$ by $U^2(.) = \varphi(U^1(.))$. Deviating from the notation elsewhere in this paper, all outcomes are expressed in $U^1$ units in this proof. It means that we replace $I$ by $U^1(I)$ which again is a nonpoint interval, that $U^1$ is linear, and $U^2 = \varphi$. Eq. (4) can be rewritten as $y_j = P^1(A)x_j + (1 - P^1(A))z_j$. Regarding necessity of the preference condition, if $\varphi$ is concave then, by Corollary 4, $\succcurlyeq^2$ is convex w.r.t. outcome mixing, which, given the equality just rewritten, implies Eq. (5) and, hence, that $\succcurlyeq^2$ is more outcome-risk averse than $\succcurlyeq^1$. The rest of this proof concerns sufficiency. We assume that $\succcurlyeq^2$ is more outcome-risk averse than $\succcurlyeq^1$ and derive concavity of $\varphi$.

There exists a nondegenerate event $A$ for $\succcurlyeq^1$, and a nondegenerate event $B$ for $\succcurlyeq^2$. From now on we only consider binary acts $x_{1_A}x_2$ for $\succcurlyeq^1$ and $x_{1_B}x_2$ for $\succcurlyeq^2$, denoting them $(x_1, x_2)$ (no confusion will arise). Define $0 < q = P^1(A) < 1$. Consider any $x = (x_1, x_2) \succcurlyeq^2 (z_1, z_2) = z$. To derive convexity of $\succcurlyeq^2$ on the set of binary acts considered here, we have to show:

$$\forall\, 0 < \lambda < 1 : \lambda x + (1-\lambda)z \succcurlyeq^2 z. \tag{B.2}$$

We first show it for $\lambda = q$. By continuity we can decrease $x_1, x_2$ into $x_1{}', x_2{}'$ such that $(x_1{}', x_2{}') \sim^2 (z_1, z_2)$. Define $y_j = qx_j{}' + (1-q)z_j,\ j = 1, 2$. With these definitions, Eq. (4) is satisfied with $x'$ instead of $x$. By Eq. (5), $y \succcurlyeq^2 z$. By monotonicity, $qx + (1-q)z \succcurlyeq^2 y \succcurlyeq^2 z$. Eq. (B.2) holds for $\lambda = q$. By repeated application and transitivity, the equation follows for a subset of $\lambda$s dense in $[0, 1]$ and then, by continuity, for all $\lambda$. Convexity of $\succcurlyeq^2$ on the two-dimensional set of acts considered here has been proved. Concavity of $\varphi$ follows from the main Theorem 3. □

**Proof of Corollary 6.** RDU on a comoncone coincides with EU on that comoncone w.r.t. a finitely additive probability measure $P$, which is convex. Hence, concavity of $U$ implies convexity of $\succcurlyeq$ by Lemma B.1. Conversely, assume that $\succcurlyeq$ is convex. Apply the main Theorem 3 to any two-dimensional subspace $\{\alpha_E\beta \in \mathcal{F} : \alpha \geq \beta\}$ of the comoncone $SF'$ with $E$ nondegenerate, and concavity of $U$ follows. □

The following lemma is the main step in deriving implications of the main Theorem 3 for weighting functions. The inequality in the lemma states that the decision weight of $s_1$ is nondecreasing in rank. It implies the same inequality for $s_2$ and is equivalent to convexity of $W$. Showing this for higher dimensions ($n > 2$) goes the same way as for two dimensions, which is why this lemma captures the essence.

**Lemma B.2.** *Assume $n = 2$, RDU, and convexity of $\succcurlyeq$. Then $W(s_1) \leq 1 - W(s_2)$.*

**Proof.** Take an outcome in $int(I)$, 0 wlog, at which $U$ is differentiable. Wlog, $U(0) = 0$. We consider a small positive $\alpha$ tending to 0, with $o(\alpha)$, or $o_\alpha$ for short, the usual notation for a function with $lim_{\alpha\to 0}\frac{o_\alpha}{\alpha} = 0$. In other words, in first-order approximations $o_\alpha$ can be ignored. We write $\pi_1 = W(s_1)$, $\pi_2{}' = W(s_2)$.

Assume $\pi_1 > 0$ and $\pi_2{}' > 0$; otherwise we are immediately done. Because of continuity of $U$ on $int(I)$ and differentiability at 0, we can obtain, for all $\alpha$ close to 0, the left indifference in

$$(\pi_2{}'\alpha, 0) \sim (0, \pi_1\alpha + o_\alpha) \preccurlyeq (\mu\pi_2{}'\alpha, (1-\mu)(\pi_1\alpha + o_\alpha)). \tag{B.3}$$

The preference is discussed later. We compare two values: the $\mu, 1-\mu$ mixture of the RDU values (which are the same) of the left two acts and the RDU value of their $\mu, 1-\mu$ mixture, which is the right act. We take $\mu > 0$ so small that the left outcome $\mu\pi_2{}'\alpha$ in the mixture is below the right outcome. Informally, by local linearity, in a first-order approximation the only difference between the two values compared is that for the left value the left outcome $\pi_2{}'\alpha$ receives the highest-outcome decision weight $\pi_1$ whereas for the right value it receives the lowest-outcome decision weight $1 - \pi_2{}'$. Convexity of $\succcurlyeq$ implies the preference in Eq. (B.3), which implies $1 - \pi_2{}' \geq \pi_1$.

Formally, note that different appearances of $o_\alpha$ can designate different functions. Thus we can, for instance, write, for constants $k_1$ and $k_2$ independent of $\alpha$: $k_1o_\alpha + k_2o_\alpha = o_\alpha$. The following is most easily first read for linear utility, when all terms $o_\alpha$ are zero. Write $u' = U'(0)$;

$\mu$ can be chosen independently of $\alpha$. Here is the comparison of the aforementioned two values: $\mu\pi_1 u'\pi_2{}'\alpha + o_\alpha + (1-\mu)\pi_2{}'u'\pi_1\alpha + o_\alpha \leq (1-\pi_2{}')u'\mu\pi_2{}'\alpha + o_\alpha + \pi_2{}'u'(1-\mu)\pi_1\alpha + o_\alpha$. Dividing by $\mu u'\pi_2{}'\alpha$, we obtain $\pi_1 \leq 1 - \pi_2{}' + \frac{o_\alpha}{\alpha}$. Now $\pi_1 \leq 1 - \pi_2{}'$ follows. □

**Proof of the main Corollary 7.** Necessity of the preference condition follows from Lemma B.1. We, therefore, assume convexity of $\succcurlyeq$. Concavity of $U$ follows from considering any nondegenerate event $E$ and applying the main Theorem 3 to the set of acts $(E : x_1, E^c x_2)$ with $x_1 \geq x_2$. We finally derive convexity of $W$.

Assume $A, B, B'$ as in Eq. (3). Write $C = B' - B$, $R = S - (A \cup B')$. Take $\gamma > \beta \in int(I)$, and consider $\mathcal{F}^* = \{(B : \gamma, C : x_1, A : x_2, R : \beta) \in \mathcal{F} : \gamma \geq x_1 \geq \beta, \gamma \geq x_2 \geq \beta\}$. This space is isomorphic to the space $\mathcal{F}$ of Lemma B.2 and the convexity inequality needed here follows from the one of that lemma. Details are as follows. Take outcome space $I^* = [\beta, \gamma]$, $s_1^* = C$, $s_2^* = A$, and weighting function $W^*(E) = \frac{W(E \cup B) - W(B)}{W(A \cup B \cup C) - W(B)}$. (If the denominator is 0, then the convexity inequality is trivially satisfied.) The inequality $W^*(\{s_1, s_2\}) - W^*(s_1) \geq W^*(s_2)$ in Lemma B.2 is the same as the required $W(A \cup B') - W(B') \geq W(A \cup B) - W(B)$. □

**Proof of Corollary 8.** Statement (i) follows from the main Corollary 7 because convexity of $W$ on every $\mathcal{F}_E$ is equivalent to subadditivity. Statement (ii) follows immediately from the main Theorem 3. □

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