A note on model VCSP1

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In Freling, Huisman and Wagelmans [1] we propose a mathematical formulation for the VCSP for the general case with two minor assumptions, namely that (**1**) there is continuous attendance (there is always a crew present if a vehicle is outside the depot) and (**2**) a short arc consists of only one dh-task. In this note we will look at the model for the general case without these assumptions.

Before providing the mathematical formulation, we need to introduce some notation. N denotes the set of trips, K denotes the set of duties, and $A^s \subset A$ and $A^l \subset A$ denote the sets of short and long arcs, respectively. We assume that deadheads to and from the depot correspond to one dh-task each. Suppose $(i,j) \in A^l$ and let el_i and bl_j denote the ending and starting location of trips i and j, respectively. Then we let $K(i,t)$ and $K(s,j)$ denote the dh-task from el_i to the depot and from the depot to bl_j, respectively. Furthermore, I_1 denotes the set of trip tasks and $K(p)$ is the set of duties covering trip task $p \in I_1$. $K(i,j)$ denotes the set of dh-tasks corresponding to deadhead $(i,j) \in A^s$. Decision variables y_{ij} and x_k are defined as before, that is, y_{ij} indicates whether a vehicle covers trip j directly after trip i or not, while x_k indicates whether duty k is selected in the solution or not. The VCSP can be formulated as follows.

(VCSP1):

$$\min \sum_{(i,j)\in A} c_{ij} y_{ij} + \sum_{k \in K} d_k x_k \tag{1}$$

$$\sum_{\{j:(i,j)\in A\}} y_{ij} = 1 \qquad \forall i \in N, \tag{2}$$

$$\sum_{\{i:(i,j)\in A\}} y_{ij} = 1 \qquad \forall j \in N, \tag{3}$$

$$\sum_{k \in K(p)} x_k = 1 \qquad \forall p \in I_1, \tag{4}$$

$$\sum_{k \in K(i,j)} x_k - y_{ij} = 0 \qquad \forall (i,j) \in A^s, \tag{5}$$

$$\sum_{k \in K(i,t)} x_k - y_{it} - \sum_{\{j:(i,j) \in A^l\}} y_{ij} = 0 \qquad \forall i \in N, \tag{6}$$

$$\sum_{k \in K(s,j)} x_k - y_{sj} - \sum_{\{i:(i,j) \in A^l\}} y_{ij} = 0 \qquad \forall j \in N, \tag{7}$$

$$x_k, y_{ij} \in \{0,1\} \qquad \forall k \in K, \forall (i,j) \in A. \tag{8}$$

The objective coefficients c_{ij} and d_k denote the vehicle cost of arc $(i,j) \in A$, and the crew cost of duty $k \in K$, respectively. The objective is to minimize the sum of total vehicle and crew costs. The first two sets of constraints, (2) and (3), correspond to the quasi-assignment formulation for the SDVSP. Constraints (4) assure that each trip task p will be covered by one duty in the set $K(p)$. Furthermore, constraints (5), (6) and (7) guarantee the link between dh-tasks and deadheads in the solution, where deadheads corresponding to short and long arcs in A are considered separately. In particular, constraints (5) guarantee that each deadhead from i to j is covered by a duty in the set $K(i,j)$ if and only if the corresponding short arc is in the vehicle solution. The other two constraint sets, (6) and (7), ensure that the dh-tasks from el_i to t and from s to bl_j, possibly corresponding to long arc $(i,j) \in A$, are both covered by one duty if and only if the corresponding deadheads are in the solution. Note that the structure of these last three sets of constraints is such that each constraint corresponds to the possible selection of one duty from a large set of duties, where the fact whether or not a duty has to be selected depends on the values of the corresponding y variables.

If the assumptions (**1**) and (**2**) do not hold we need to replace constraints (5) by the following set of constraints:

$$\sum_{k \in K(i,j)} x_k - y_{ij} \geq 0 \quad \forall (i,j) \in A^s. \tag{9}$$

We explain the correctness of this formulation for both cases with an example.

In the first case we have no continuous attendance. Suppose we have the following two duties:

duty 1: $s \to 1 \to 2 \to 3 \to 4 \to t$,

duty 2: $s \to 5 \to 6 \to 7 \to 8 \to t$,

where the odd numbers correspond to the start points of the four trips, called i, j, k and l and the even numbers correspond to the end points of the same trips. The break in the first duty is between 2 and 3 and in the second duty between 6 and 7. If duty 1 is selected, then it is allowed that there is a vehicle doing trip j or trip l immediately after trip i. So if $x_1 = 1$, then y_{ij} or y_{il} is 1. Constraints (2) and (3) guarantee that every trip has one successor and one predecessor, so we know that y_{ij} and y_{il} are not both equal to 1. This is the reason why equality in the constraints (5) is not valid and we need to have constraints (9). For this example, we then get the following constraints:

$$\begin{aligned} x_1 - y_{ij} &\geq 0 \\ x_1 - y_{il} &\geq 0 \\ x_2 - y_{kl} &\geq 0 \\ x_2 - y_{kj} &\geq 0 \end{aligned}$$

In the other case, a short arc consists of more than one dh-task. Suppose that we have one short arc that is corresponding to two dh-tasks. For example, we have the following five duties and again four trips:

duty 1: $s \to 1 \to 2 \to 7 \to 3 \to 4 \to t$,
duty 2: $s \to 7 \to 8 \to t$,
duty 3: $7 \to 8 \to t$,
duty 4: $s \to 5 \to 6 \to 3$,
duty 5: $s \to 5 \to 6 \to t$,

where the numbers are corresponding to the start and end points of the trips i, j , k and l as before. Only in the first duty there is a break between 7 and 3. The short arc (2,3) consists of two dh-tasks, namely (2,7) and (7,3). This means that if duty 1 is selected, a vehicle can do trip l (starts with 7) or trip j (starts with 3) directly after trip i (ends with 2). Therefore we cannot have equality in the constraint set (5), but we have the following constraints corresponding to set (9):

$$\begin{aligned} x_1 - y_{ij} &\geq 0 \\ x_1 - y_{il} &\geq 0 \\ x_4 - y_{kj} &\geq 0 \end{aligned}$$

There are two feasible solutions, namely duty 1, 2 and 5 with three vehicles, where every vehicle does exactly the same as the three duties and duty 1, 3 and 4 with two vehicles, where there is a changeover in duty 1. It is obvious that above constraints give exactly these solutions.

Finally, we want to remark that the model with constraint (9) is also a correct model if the assumptions (**1**) and (**2**) hold, because then the formulation with (5) is the same as with (9). This is the same, because constraints (4) assure that every trip task is in exactly one duty. So in the case that both assumptions hold, we have that if a certain trip j follows directly after trip i in a selected duty, then no trip $k \neq j$ follows directly after trip i for another selected duty. Thus $y_{ik} = 0 \; \forall k \neq j$. Constraints (2) assure now that if $\sum_{k \in K(i,j)} x_k = 1$, then $y_{ij} = 1$. So the formulation with (5) is the same as with (9).

References

[1] R. Freling, D. Huisman, and A.P.M. Wagelmans. Models and algorithms for integration of vehicle and crew scheduling. Technical Report EI2000-10/A, Econometric Institute, Erasmus University Rotterdam, Rotterdam, 2000.