Signaling in a stochastic environment and dynamic limit pricing*

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Abstract

This paper studies a two-player signaling game in continuous time in which the payoff of the uninformed player depends on the publically-observed level of stochastic environment and also on the privately-observed type of the informed player. The two types of the informed player pool as long as the payoff stays within two-sided bounds. In equilibrium, the informed player reveals her type at a randomized lower trigger. The uninformed player gradually learns about the true type by observing the path of the environment and, at an upper boundary, verifies the opponent's type to claim the payoff. We apply the game to model dynamic limit pricing under stochastic demand and derive a set of inferences unavailable in one-period deterministic models.

Keywords: Signaling games; Stochastic games; Limit pricing.

1 Introduction

Many important economic situations that involve incomplete information and signaling are set in dynamic stochastic environments. A monopolist that uses prices to signal unprofitable entry, as in the limit pricing model of Milgrom and Roberts (1982), in reality has to do so repeatedly and under changing market conditions. A firm that uses dividends to signal its profitability, as in e.g. Miller and Rock (1985), typically makes payout decisions repeatedly while facing stochastic cash flows. Yet, the available models based on signaling games allow only for one-time signals or repeated signals under stationary conditions. The purpose of this paper is to extend the signaling game to a fully dynamic stochastic model.

We study a new class of two-player signaling games in continuous time in which the stake contested by the uninformed player is a diffusion process X_t observed by both players. The informed player's type is either strong or weak and initially her type is only privately observed.

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The uninformed player obtains the stake if he contests it from the weak type, but receives nothing if the other player is strong. The informed player of the weak type gets a negative payoff if contested, while the strong type is unaffected by the contestant. It is possible, but costly, for the informed player to send signals and keep the other player uninformed about her own true strength.

We focus here on how costly signaling in pooling and semi-pooling outcomes can be dynamically sustained in favorable and unfavorable situations. How does the informed player signal in such a setting? How does the uninformed player make the strategic decision to contest the stake? Our primary insight is that, as the stake evolves in a stochastic environment, at some point in time the incentive constraints may stop being binding. In particular, provided that the game starts at a pooling situation, the uninformed player wants to contest as the stake gets high enough given his belief about the other player's type. On the other hand, if the stake gets low enough, there is little threat from the contestant, so the informed player of the weak type does not want to send costly signals anymore and, consequently, prefers to reveal her type.

The decisions of the two players are strategically interrelated. The best responses are optimal stopping decisions, with the uninformed (respectively informed) player seeking a critical high state U (respectively low state L) to stop the signaling game. The equilibrium strategy U of the uninformed player balances the benefits of winning the stake with some current belief that the type is weak and the cost of loosing the opportunity to learn the revealed type if the stake reaches L. The strategy L of the weak informed player strikes the balance between the benefits of facing the uninformed contestant as late as U but bearing the signaling cost and the benefits of revealing her type, and not paying the signaling cost but facing early entry of the contestant that is sure to win the stake.

We show that the stopping game in general has no Markov perfect equilibrium in pure strategies. Instead, in equilibrium the informed player reveals her type at a randomized lower trigger. The reason for this is that a deviation from a pure strategy L provides a discrete gain while it bears an infinitesimal cost. Specifically, suppose that L is an equilibrium pure strategy of the weak player. If no action is taken at L, the uninformed player updates his belief and is certain to face the strong informed player. Then a slight deviation from L discretely improves the standing of the informed player against the uninformed one, while the cost of additional signaling is, due to continuous time, infinitesimal.

The mixed type-revelation strategy introduces a remarkable Bayesian learning process based on the path of the stake X. The uninformed player, observing no actions in the support of the strategy of the informed player, updates his belief about the true type. Therefore the minimum process $M_t = \min_{0 \leq s \leq t} X_s$ can be used as a state variable governing the belief process. Based on these observations, the uninformed player's problem is non-trivial and involves path-dependent payoffs and learning from the path of the diffusion process. To solve it we use some recent developments in the theory of optimal control of extremum processes. Our characterization of the Markov perfect Bayesian equilibrium is relatively basic and is a solution to two ordinary differential equations subject to boundary conditions. In particular we show that in equilibrium the weak informed player reveals at a mixed-strategy lower trigger that is continuously distributed over some interval in X. The uninformed player contests the stake at an upper boundary that is decreasing in the running minimum on the same interval.

Two extensions from the standard one-period (two-dates) signaling models generate the interesting strategic interactions with learning in our model. These are multi-period dynamics and uncertainty about the future stake. In a one-shot game with a stochastic stake, there is no room for waiting to obtain information in the future. On the other hand, in a multi-period game with a fixed stake (or varying in time but deterministic and monotone), the strategic situation is non-trivial only at the initial node. Beyond the initial date, no learning and no type revelation can happen. Consequently, similar results to the ones presented in this chapter can be obtained in other setups that have multi-period signaling in stochastic environment. We choose for the continuous time framework which is standard and tractable in studies of timing and stopping games (e.g. Fudenberg and Tirole (1986), Bulow and Klemperer (1999), Dutta and Rustichini (1993)). To incorporate a stochastic environment in a tractable way we model the stake as a geometric Brownian motion.

By introducing continuous-time dynamics and uncertainty, our model is able to provide new insights into some of the well-known signaling situations in economics. As an illustration we apply the generic model to entry deterrence by limit pricing. We can translate our setting in a limit pricing problem by interpreting the diffusion process as stochastic demand, the informed player as the incumbent firm and the uninformed as a potential entrant. One advantage of our setup is that we can explore equilibrium price dynamics. The stochastic limit-pricing game implies that price dynamics may reveal limit pricing of incumbents. Specifically, in equilibrium the limit-pricing incumbent reveals its type by increasing prices as the market conditions get unfavorable to entry. To an external observer this means that increasing prices in a decreasing market may be interpreted as an indicator of entry deterring limit pricing. This observation brings forward a policy instrument to detect anti-competitive pricing practices. This is in contrast to the standard one-shot signaling models of limit pricing that provide little in terms of antitrust policy recommendations. We also show that the decision of the entrant to enter exhibits path dependence. Despite the fact that the demand is modeled as a Markovian variable, the entrant assessment of entry profitability depends not only on the current market, but also on the historical minimum. Moreover, the model implies that the learning mechanism postpones entry. In a dynamic stochastic environment, the well-known effect of the value of waiting delays entry decisions: the entrant waits to observe future demand realizations. However, in our setup the entrant firm delays entry decision even further in anticipation of possible learning about the incumbent type.

A few previous studies consider dynamic aspects of signaling. Saloner (1984) presents a multi-period version of the limit-pricing model of Matthews and Mirman (1983) in which signals received by the uniformed firm are noisy. In contrast, we assume that actions of the informed player are observed directly by the uniformed party. The key difference of our model is that we allow for a stochastic environment that changes over time. This means that there are states when signaling is more profitable and times when it is not profitable. In the paper of Saloner (1984), demand is uncertain, but demand shocks, that last for a single period, serve solely as a device to add noise to the informed player's actions; the market conditions for both payers are identical before each round. Mester (1992) analyzes a three-period signaling setting in which the unobservable type changes over time. Toxvaerd (2007) adapts and extends a similar setup to study limit pricing. Contrary to these papers, we assume that the *observable* market conditions fluctuate but the unobservable type of the informed player is fixed. In our setting we analyze richer dynamics of the stochastic variable and effects of good and bad states on signaling strategies. Kaya (2008) studies separating equilibria in an infinitely repeated discrete-time signaling game, while we concentrate on how pooling and semi-pooling equilibria can be sustained in a stochastic environment. Additionally, none of these models are formulated in continuous time.

Our work relates also to the literature on continuous-time games with an underlying diffusion process and especially to some recent papers that use continuous-time methods to solve dynamic information problems. Sannikov (2007) and Sannikov and Skrzypacz (2007) study repeated games with imperfect monitoring in continuous time. Sannikov (2008) analyzes a repeated moral hazard problem in continuous time. Our general setup is also related to some strategic models of investment under uncertainty. Lambrecht and Perraudin (2003) study investment decisions in a stochastic environment under asymmetric information. In their model both players are uninformed about the other's type, but there is no signaling allowed and the strategic situation is a preemption game. Décamps and Mariotti (2004) study an investment problem in duopoly with incomplete information, but also here signaling is not allowed and the game is a war of attrition. Dutta and Rustichini (1995) analyze a different stochastic game to ours, but also there two players control the two sides of a diffusion process. Despite some similarities, neither of these previous models analyze a signaling game, and their sets of strategic interactions are very different to ours.

The next section sets up the model. In Section 3 we study special cases with complete information and a deterministic environment. Section 4 presents the equilibrium analysis in the stochastic model. In Section 5 we apply the general signaling model to analyze limit pricing under stochastic demand. Section 6 concludes, and an Appendix collects the proofs omitted in the main text.

2 Model

2.1 Setup

The game is set in continuous time with infinite horizon, indexed with $t \in [0, \infty)$. There are two players. Player 1 is of type $\theta \in \{w, s\}$ (weak or strong) and knows her type. If Player 1 does not make any effort, θ is observed by Player 2. By exercising some costly effort, Player 1 of the weak type can mimic the behavior or appearance of the strong type and in this way pretend to be of type s. The cost of imitation per unit of time is $c > 0$. Player 2 has a prior belief $\pi_0 \in (0,1)$ that $\theta = w$. Observing Players 1's actions, Player 2 updates his belief about θ using Bayes rule whenever possible and the belief at time t is denoted by π_t. In particular, at the first time the w-type ceases to send the signal, the belief is updated to 1.

Player 2 contests the stake of the game which exogenously evolves over time according to a geometric Brownian motion

$$dX_t = \mu X_t dt + \sigma X_t dZ_t,$$

with $X_0 = x_0$. The constants μ and $\sigma > 0$ are drift and volatility parameters. Z_t is a standard Brownian motion. At any (stopping) time $t \geq 0$, Player 2 can contest the stake by paying a fee of $K > 0$ (K could be interpreted as an entry cost or a checking fee; throughout the paper we refer to Player 2 contesting the stake also as entering or stopping). At this time, Player 2 learns θ and gets a payoff X_t if $\theta = w$ and a zero payoff if $\theta = s$. When checked by Player 2, Player 1 gets a fixed negative payoff normalized to -1 if she is of type w and 0 if she is of type s.

Both players are risk neutral and discount flows at a constant discount rate r. To concentrate on the interesting cases we assume throughout the paper that $c < r$, which is a necessary condition for Player 1 of the w type to have an incentive to use signals to postpone entry. Similarly, we assume $\mu < r$ to guarantee convergence of the problem of Player 2.

Some aspects of our modeling strategy deserve comment. The choice of the particular payoff functions stems from our objective to keep the analysis simple and to incorporate the following desired features of the game. The environment X is stochastic and the payoffs depend on the state of the environment. In particular, the uninformed player wants to contest if the state is 'good' (high X). The uninformed player is worse off against the strong type of the informed player. Absent the cost of signaling, the weak type prefers to be recognized as a strong one. Finally, signaling is costly. We do not aim here to show the most general functional forms that support our results, but instead to analyze a simple game iwth desired characteristics. Certainly our analysis can accommodate other payoff structures that preserve the above mentioned features. Indeed, we consider an tractable example of limit pricing in Section 5 where payoffs are in flows and the signaling cost and the informed player's payoff depend directly on the state X.

As discussed above, our focus in this paper is to study how costly signaling in pooling and semi-pooling outcomes can be dynamically sustained in good and bad states. To do so in an economic manner, we model signaling in a reduced way and we limit the room for separating strategies. It is sufficient for our purpose not to explicitly model the costly signals used by the w type. The implicit assumption is that the pooling strategy in equilibrium is the efficient one, i.e. the one least costly to Player 1.

A further simplifying assumption is that the s type is infinitely strong and is indifferent to entry. In this way we leave aside discussion of separating strategies. Our analysis conveys to the case in which the s type has little incentives to separate (i.e. the s type looses little when contested or when the cost of separation is high) and therefore does not separate (we indeed assume that the s type is less than infinitely strong in Section 4 to prove uniqueness of equilibrium strategies). Another important situation to which our model applies directly is

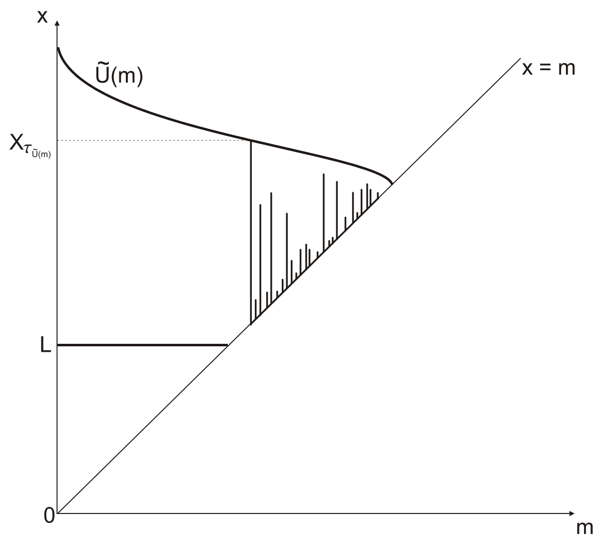

Figure 1: A sample path of (x, m).

when the signaling space is limited. The sharpest, yet frequently realistic, case is the binary signaling space which leaves little room for separating strategies. For concreteness one could interpret our baseline model as a game with a binary signaling space. Finally, we note that in any case a separating equilibrium would be less remarkable in our setup as it would eliminate the interesting belief dynamics. Dynamic separating strategies have been convincingly studied in the recent paper of Kaya (2008) and a stochastic environment would not significantly effect that analysis.

2.2 State space, strategies and Markov equilibria

In principle, the state space of payoff relevant variables consists of a pair of Markov state variables $(x, \pi) \in \mathbb{R}_+ \times [0, 1]$. It will simplify our analysis if we transform the state space based on the following important observation. As we will show in Section 4.2, in the continuation region, before the game is stopped, the belief variable π_t is a function of the running minimum of X_t, that is of $M_t = \min_{0 \leq s \leq t} X_s$. The intuition is that, as Player 1 may prefer to reveal her type as the stake becomes low, the historical minimum may be used by Player 2 to update his belief about θ. At most parts of the analysis, it will be more convenient to work with the minimum than with the beliefs so, where indicated, we analyze the game in the state space $\left\{(x, m) \in \mathbb{R}^2_+ : x \geq m\right\}$.

Player 1 of the weak type takes an action $a_1 \in \{signal, reveal\}$, where $a_1 = reveal$ indicates a decision to stop signaling and to reveal the player's type. As the strong type is passive, in the sequel we shall implicitly mean the weak type, when we discuss actions and strategies of Player 1. Once the type is revealed, the game becomes a game of complete information with no strategic interactions. Given optimal behavior of the players beyond this point, the expected discounted payoffs are considered as the terminal payoffs of the signaling game. To specify these termination payoffs suppose that the w type reveals at time t and denote by $\tau_w \geq t$

Threshold		$x = L(\pi)$	$x = U(\pi)$
Stopping time		τ_L	τ_U
Player 1	w type	$\Omega = E[-e^{-r(\tau_w-\tau_L)}]$	-1
	s type	$-$	0
Player 2		$W = E[e^{-r(\tau_w-\tau_L)}(U(1) - K)]$	$\pi_{\tau_U} U(\pi_{\tau_U}) - K$

Table 1: Terminal payoffs.

the (stopping) time at which Player 2 would optimally collect the stake upon the payment of K. Then the expected discounted payoff at time t of Player 2 is $E[e^{-r(\tau_w-t)}\,(X_{\tau_w} - K)]$ and of Player 1 is $E[-e^{-r(\tau_w-t)}]$. Player 2 takes an action $a_2 \in \{\textit{do not contest}, \textit{contest}\}$. Once $a_2 = \textit{contest}$ is chosen, the signaling game is over and the terminal payoffs are collected.

Markov strategies prescribe actions to the current state. In the signaling game, Markov strategies define two sets in the state space, a continuation set ($a_1 = \textit{signal}$ and $a_2 = \textit{do not contest}$) and a stopping set ($a_1 = \textit{reveal}$ and $a_2 = \textit{contest}$). In other words, each player faces an optimal stopping problem. The theory of optimal stopping of Markov process indicates that the strategies will take the form of an optimal stopping boundary (see Peskir and Shiryaev (2006)). In the case of Player 2, define $U : [0, 1] \to \mathbb{R}_+$, then $U(\pi)$ is a *boundary* separating continuation and stopping regions in the state space (x, π). Precisely, Player 2 chooses '*do not contest*' if $x < U(\pi)$ and chooses '*contest*' whenever $x \geq U(\pi)$. The associated stopping time is defined as $\tau_U = \inf\{\tau \geq 0 : X_\tau \geq U(\pi)\}$. Analogously, if the strategy of Player 2 is considered in the state space (x, m), the function $\tilde{U} : \mathbb{R}_+ \to \mathbb{R}_+$ is a free boundary such that Player 2 '*does not contest*' if $x < \tilde{U}(m)$ and '*contests*' whenever $x \geq \tilde{U}(m)$.

A solution to the problem of Player 1 takes the form of a lower boundary $L(\pi)$. Player 1 signals as long as $x > L(\pi)$ and reveals as soon as $x \leq L(\pi)$. Formally, the strategy prescribes a stopping time $\tau_L = \inf\{\tau \geq 0 : X_\tau \leq L(\pi)\}$. The stopping time at which Player 2 enters after the type is revealed denoted above as τ_w is equal to $\inf\{\tau \geq \tau_L : X_\tau \geq U(1)\}$.

As an illustration of the two-dimensional process (x, m), Figure 1 presents a sample path in the continuation region between some L (here constant) and $\tilde{U}(m)$ (here decreasing in m) with a realization of the stopping rule at X_{τ_U}. Above the diagonal $x = m$, the process evolves vertically reflecting changes in x. When new minima are reached, the process moves down along the diagonal. Since the process stays above L and reaches first $\tilde{U}(m)$, here Player 2 contests before Player 1 has revealed her type.[1]

Table 1 collects the information about the terminal payoffs when the signaling game is stopped at either $L(\pi)$ or $U(\pi)$. Given a pair of (pure) strategies (L, U) and the starting values

[1] Note that Figure 1, which aims to present the (x, m) process, does not necessarily represent any equilibrium or sensible strategies. In fact in a game in pure strategies, starting at π_0, the belief can be only updated to either 0 or 1 and the signaling game is stopped at either $L(\pi_0)$ or $U(\pi_0)$ and thus $\tilde{U}(m)$ is constant for $m > L(\pi_0)$.

of x and π, the respective total expected payoffs of Player 1 and 2 are given by

$$\begin{aligned} R_1(x,\pi;L,U) &= E_{x,\pi}\left[-\int_0^{\tau_L\wedge\tau_U} e^{-rt}c\,dt - e^{-r\tau_U}1_{\tau_L\geq\tau_U} - e^{-r\tau_w}1_{\tau_L<\tau_U}\right], \\ R_2(x,\pi;L,U) &= E_{x,\pi}\left[e^{-r\tau_U}\left(\pi_{\tau_U}U(\pi_{\tau_U})-K\right)1_{\tau_L\geq\tau_U} + e^{-r\tau_w}(U(1)-K)1_{\tau_L<\tau_U}\right], \end{aligned}$$

where by $E_{x,\pi}[\cdot]$ we denote the conditional expectation $E[\cdot|(X_0,\pi_0)=(x,\pi)]$.

It is essential for our analysis to allow the players to randomize across pure strategies. As we show later, in the general case the game has no perfect equilibrium in pure strategies. As our subsequent analysis focuses on the case in which Player 1 applies a mixed strategy and Player 2 responds with a pure strategy, we need to consider only a mixed strategy of Player 1. A *mixed strategy* of Player 1 is a probability measure P on $[0,x_0]$ with the corresponding distribution function G defined by $G(x)=P([x,x_0])$.[2] G is interpreted as a distribution function over trigger strategies. The expected payoff that corresponds to a pair of strategies (G,U) is $R_i(x,\pi;G,U)=\int R_i(x,\pi;L,U)dG$ for $i=1,2$.

A *Markov perfect Bayesian equilibrium* (MPBE) is a pair of Markov strategies (G^*,U^*) such that

$$\begin{aligned} R_1(x,\pi;G^*,U^*) &\geq R_1(x,\pi;G,U^*), \\ R_2(x,\pi;G^*,U^*) &\geq R_2(x,\pi;G^*,U), \end{aligned}$$

for all states (x,π) and all strategies G and U.

3 Simple cases

3.1 Complete information

Suppose that for some $t\geq 0$ $\pi_t\in\{0,1\}$, so that signaling does not play a role in the game. If $\pi_t=0$, i.e. Player 1 is strong with probability one, then obviously Player 2 never tries to contest the stake, i.e. $U(0)=\infty$. If $\pi_t=1$, then Player 2 solves the optimal stopping problem

$$W(x)=\sup_{t\leq\tau\leq\infty} E\left[e^{-r(\tau-t)}\left(X_\tau-K\right)|X_t=x\right].$$

As usual, the optimal strategy in the solution of the problem takes the form of an upper trigger. Let $U(1)$ denote the stopping threshold. By the standard dynamic programming argument and Itô's lemma, $W(x)$ satisfies the following Hamilton-Jacobi-Bellman differential equation

$$rW(x)=\mu xW'(x)+\frac{1}{2}\sigma^2x^2W''(x) \tag{1}$$

[2] Our definition follows the logic of mixed strategy, i.e. a player chooses a randomized pure strategy, which in our case is a stopping time. More detailed discussion of mixed and behavioral strategies and their equivalence in stopping games can be found in Touzi and Vieille (2002).

in the continuation region, i.e. for $x \in (0, U(1))$. The left-hand side of (1) reflects the required rate of return per unit of time for holding the option to get x. The right-hand side is the expected change in the value.

The differential equation is associated with the following three boundary conditions:

$$W(U(1)) = U(1) - K, \tag{2}$$

$$W'(U(1)) = 1, \tag{3}$$

$$W(0) = 0. \tag{4}$$

The value matching (2) and smooth pasting (3) conditions impose a continuous and smooth fit at the boundary, required for optimality. Condition (4) ensures that the stake will be worthless if x reaches its absorbing barrier zero. Solving equations (1)-(4) we obtain that Player 2 optimally contests the stake at the stopping time

$$\tau_w = \inf \{ \tau \geq t : X_\tau \geq U(1) \},$$

where

$$U(1) = \frac{\beta_1 K}{\beta_1 - 1}, \tag{5}$$

and

$$\beta_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2} \right)^2 + \frac{2r}{\sigma^2}}.$$

β_1 is the positive root of the characteristic equation $\frac{1}{2}\sigma^2 \beta (\beta - 1) + \mu\beta - r = 0$ and is always larger than 1. The value $W(x)$ is given by[3]

$$W(x) = \left(\frac{x}{U(1)} \right)^{\beta_1} (U(1) - K). \tag{6}$$

Knowing the optimal behavior of Player 2, let Ω denote the expected discounted value of the weak type of Player 1 if the type is revealed at time t. Then $\Omega(x) = E\left[-e^{-r(\tau_w - t)} | X_t = x \right]$. Using similar methods as above we obtain that

$$\Omega(x) = -\left(\frac{x}{U(1)} \right)^{\beta_1}. \tag{7}$$

$\Omega(L)$ and $W(L)$ are the respective terminal payoffs if Player 1's type is revealed at threshold L (see column $x = L$ in Table 1).

[3] The intuition for this and some other similar expressions in the paper can be gained by observing that $E[e^{-r(\tau - t)} | X_t = x] = (x/X_\tau)^{\beta_1}$, where τ is a stopping time. Thus, given that the current state is x, $(x/X_\tau)^{\beta_1}$ is the present value of one dollar received at stopping time τ.

3.2 Deterministic X_t

Suppose now that $\sigma = 0$, so that the process $dX_t = \mu X_t dt$ is deterministic. In this case, the strategies in the unique equilibrium differ depending on the sign of μ, but they share a similar simple structure. For any given belief π, Player 2 contests the stake as he would do in a non-strategic situation. Given this behavior of Player 2, Player 1 applies a straight-forward incentive constraint to decide between signaling or revealing (or randomizing between these two).

We first analyze the simpler case in which x decreases in time. If $\mu < 0$ then, for any fixed π, Player 2 would not postpone the entry decision and would rather enter immediately, if at all. If x falls below K, Player 2 never enters and so Player 1 does not signal.

Proposition 1 *If $\sigma = 0$ and $\mu < 0$, the signaling game has a unique MPBE. In the equilibrium Player 2 applies the following upper boundary strategy*

$$U(\pi) = \frac{K}{\pi}.$$

The lower trigger strategy of Player 1 of the w type is

$$L = \begin{cases} x_0 & if\ \Gamma_1(x_0) \leq 0, \\ K & if\ \Gamma_1(x_0) > 0, \end{cases}$$

where

$$\Gamma_1(x) = -\frac{c}{r} + \left(\frac{x}{K}\right)^{\frac{r}{\mu}} \frac{c}{r} + 1.$$

$\Gamma_1(x_0)$ characterizes the incentive compatibility constraint of Player 1 for signaling from $t = 0$ until the time when x reaches K.[4] It is simply the difference between the total costs of signaling and of revealing her type immediately in the situation that this would trigger immediate entry of Player 2.

If $\mu > 0$ and x increases in time, several additional considerations arise. First, for any $\pi > 0$ there is a sufficiently large x such that Player 2 decides to contest the stake. Second, apart from the full separation and pooling outcomes arising also in the case of negative μ, if $\mu > 0$ there is an intermediate set of parameters at which the unique outcome is semi-pooling. Third, agents take into account the value of waiting (similar to the stochastic, complete information case of Section 3.1).

Proposition 2 *If $\sigma = 0$ and $\mu > 0$, the signaling game has a unique MPBE. In the equilibrium Player 2 applies the following upper boundary strategy*

$$U(\pi) = \frac{K}{\pi} \frac{r}{r - \mu}.$$

[4] Note that the weak and strict inequalities in the strategy L are rather arbitrary, but the events with equality are of measure zero.

The lower trigger strategy of Player 1 of the w type is

$$L = \begin{cases} x_0 & if\ \Gamma_3(x_0) \leq 0, \\ \begin{cases} x_0 & with\ prob.\ p \\ 0 & with\ prob.\ (1-p) \end{cases} & if\ \Gamma_2(x_0) < 0 < \Gamma_3(x_0), \\ 0 & if\ \Gamma_2(x_0) \geq 0, \end{cases}$$

where

$$\begin{aligned} \Gamma_2(x) &= -\frac{c}{r} - \left(\frac{x}{U(\pi_0)}\right)^{\frac{r}{\mu}} \left(1 - \frac{c}{r}\right) + \left(\frac{x}{U(1)}\right)^{\frac{r}{\mu}}, \\ \Gamma_3(x) &= -\frac{c}{r} + \left(\frac{x}{U(1)}\right)^{\frac{r}{\mu}}, \end{aligned}$$

and

$$p = \frac{1 - N\pi_0}{(1-N)\pi_0},$$

with

$$N = \frac{r-\mu}{r}\frac{x_0}{K}\left[\frac{r-c}{r\left(\frac{x_0}{U(1)}\right)^{\frac{r}{\mu}} - c}\right]^{\frac{\mu}{r}}.$$

$\Gamma_2(x)$ and $\Gamma_3(x)$ characterize the incentive constraints of Player 1. $\Gamma_2(x_0)$ is the difference between the total costs of signaling with entry at $x = U(\pi_0)$ and of revealing immediately with entry at $x = U(1)$. $\Gamma_3(x_0)$ is the difference between the costs of continuous signaling without entry and of revealing immediately.

The reason that the w type randomizes at $t = 0$ for some intermediate parameter values is intuitively clear. Because x increases over time, Player 2 ultimately contests the stake under a pooling outcome. Taking this into account Player 1 does not choose (a pure strategy) signaling if $\Gamma_2(x_0) < 0$. Under the full separating outcome Player 2 never enters if he expects to face the s type. Then, however, the w type may be tempted to deviate and imitate the s type if $\Gamma_3(x_0) > 0$. As $\Gamma_2(x_0) < \Gamma_3(x_0)$, there are parameter values at which neither pure pooling nor pure separating is an equilibrium outcome. The probabilities p and $(1-p)$ follow from Bayes rule and the indifference of Player 1 for signaling and revealing at $t = 0$.

Finally, we look at the simplest case when X_t is constant over time, i.e. $\mu = \sigma = 0$. The signaling game simplifies to a repeated game under stationary conditions. It is not difficult to derive the following result as the middle ground of the two cases described in Propositions 1 and 2.

Corollary 3 *If $\mu = \sigma = 0$, the unique MPBE is*
(i) if $x_0 \leq K$, Player 1 of the w type immediately reveals and Player 2 never enters;
(ii) if $\pi_0 x_0 \leq K < x_0$, Player 1 never reveals and Player 2 never enters;
(iii) in the remaining case, Player 1 does not signal and Player 2 enters immediately against the w type and never enters against the s type.

4 Equilibrium analysis

4.1 Preliminaries

We begin with pointing out two key implications of the stochastic state variable to the signaling game. Let us denote by $\Gamma(x)$ the difference between the payoffs of the w type from continuous signaling (up to entry at $U(\pi)$ when Player 1 incurs a loss of -1) and from revealing immediately (that triggers entry at $U(1)$), that is

$$\Gamma(x) = -\frac{c}{r} + \left(\frac{x}{U(\pi)}\right)^{\beta_1}\left(\frac{c}{r} - 1\right) + \left(\frac{x}{U(1)}\right)^{\beta_1}.$$

For any $x \leq U(\pi)$, $\Gamma(x)$ captures the incentive compatibility constraint of the w type for signaling.[5] Analyzing the expression we observe that as long as $U(\pi) \geq U(1)$ then $\Gamma'(x) > 0$ for all $x \geq 0$, and $\Gamma(y) = 0$ for some $y > 0$ (the condition $U(\pi) \geq U(1)$ intuitively holds, and we shall see later it is always true in equilibrium). The implication is that if Player's 1 incentive constraint for signaling is satisfied at $t = 0$, then it will be binding whenever x exceeds x_0. However if at x falls sufficiently below x_0, then Player 1 of the w type might prefer to stop signaling, reveal her type, and wait for Player 2 to take his prize at $U(1)$. The fact that in the case $\sigma = 0$ and $\mu \geq 0$ the incentive constraint remains satisfied if it is satisfied at x_0, made the equilibrium strategies in the deterministic case relatively simple. Now, however, in the general case, the weak type of Player 1 decides when to stop signaling at lower trigger L taking into account its strategic effect on the entry decision of Player 2. Similarly, Player 2's choice of U is strategic with respect to Player 1's revelation decision.

The second key feature of the stochastic case is that, except for some extreme starting values, the signaling game has no MPBE in pure strategies.

Lemma 4 (No pure strategy equilibrium) *If the game is not stopped at $t = 0$, then there is no MPBE in pure strategies.*

The lemma is a special case of Lemma 8 below, hence we postpone the formal proof to that point. The intuition for the result is as follows. Player 2 seeing no action at the supposed equilibrium pure strategy L, updates his belief to 0 and never contests ($U(0) = \infty$). But then Player 1 of the weak type deviating from the equilibrium (not revealing at L) obtains a discrete increase in the value while bearing—due to continuous time—infinitesimal signaling cost, which upsets the proposed equilibrium.[6]

A similar intuition that explains the nonexistence of a pure strategy equilibrium leads to the anticipation that the distribution G for the mixed strategy of Player 1 should have no atoms. In the remainder of this subsection we prove in a number of steps that there are no gaps in the distribution G and no pure strategies are chosen with positive probability, except possibly at

[5] For simplicy of exposition we only consider the case that for a given (x, π), $x \leq U(\pi)$. More generally, $U(\pi)$ could be replaced by $\bar{U}(\pi) = \max\{x, U(\pi)\}$ without bringing new insights but complicating notation.

[6] A similar effect that information-revealing actions are played in a mixed strategy is also present in Huddart, Hughes and Levine (2001) in a context of informed trading.

x_0. The outline of the following analysis resembles some analyses of mixed strategies found in deterministic timing games, however the stochastic environment requires more involved proofs.

Let us denote the support of a distribution G by $\mathrm{supp}(G)$ (that is, the smallest closed set such that the distribution G assigns zero probability to all events not in this set). The question we ask first is what $\tilde{U}(m)$ must be, so that Player 1 chooses m in the support of G. We use the requirement that if $m \in \mathrm{supp}(G)$, then Player 1 must be indifferent to revealing when the minimum m is reached for the first time.

Let $F(x, m)$ denote the expected discounted value of Player 1 of the w type such that Player 1 is indifferent between stopping and continuing at $x = m$ for all $m \in \mathrm{supp}(G)$. In the continuation region, for $x \in (m, \tilde{U}(m))$ with m fixed, the following Bellman-type equation holds:

$$rF(x,m) = \mu x F_x(x,m) + \frac{1}{2}\sigma^2 x^2 F_{xx}(x,m) - c. \tag{8}$$

Note that m, the second dimension of the state space, does not appear directly in the differential equation. The reason is that m does not change during an infinitesimal time interval if $x > m$. The general solution to the differential equation is

$$F(x,m) = B_1(m)x^{\beta_1} + B_2(m)x^{\beta_2} - \frac{c}{r}, \tag{9}$$

where β_1 and β_2 are the roots of characteristic quadratic $\frac{1}{2}\sigma^2\beta(\beta-1) + \mu\beta - r = 0$, positive and negative, respectively. The continuous and smooth fit principles at the boundaries give the following conditions:

$$F(\tilde{U}(m), m) = -1, \tag{10}$$

$$F(m,m) = \Omega(m), \tag{11}$$

$$F_x(m,m) = \Omega'(m), \tag{12}$$

for all $m \in \mathrm{supp}(G)$. Condition (10) states that the continuation value at $x = \tilde{U}(m)$ equals the terminal payoff -1. $\Omega(m)$ is the payoff of the w type if the type is revealed at m and its value is given in (7). Conditions (11)-(12) reflect that Player 1 is indifferent between revealing and not revealing. Additionally the normal reflection condition[7] holds at $x = m$:

$$F_m(m,m) = 0. \tag{13}$$

Let us define $L_1 = \inf(\mathrm{supp}(G))$ to be the infimum of the set $\mathrm{supp}(G)$. The forgoing arguments can be used to characterize L_1. This is the threshold at which Player 1 optimally reveals with probability one under the most favorable circumstances, i.e. when Player 2 believes that $\theta = s$ and never enters. The differential equation (8) is then coupled with the value matching and smooth pasting conditions $F(L_1, L_1) = \Omega(L_1)$ and $F_x(L_1, L_1) = \Omega'(L_1)$. Later we

[7] The *normal reflection* conditions are used in the optimal stopping problems involving a extemum (maximum or minimum) process. For a formal verification that (8) together with a boundary condition corresponding to (13) hold in problems involving the minimum, see Peskir (1998) (see also Peskir and Shiryaev (2006, Ch. 13)).

shall also discuss the supremum L_0 of the set $\operatorname{supp}(G)$, that is $L_0 = \sup(\operatorname{supp}(G))$. Furthermore, let $\tilde{U}_G : (L_1, U(1)) \to \mathbb{R}_+$ be the solution to (14) below. The next lemma provides the condition for m to be in $\operatorname{supp}(G)$, characterizes L_1 and some properties of $\tilde{U}_G$. To shorten notation let $\delta_1(m) = (\tilde{U}(m)/m)^{\beta_1}$ and $\delta_2(m) = (\tilde{U}(m)/m)^{\beta_2}$.

Lemma 5 (Support of G) *(i) If $m \in \operatorname{supp}(G)$, then $\tilde{U}(m)$ satisfies*

$$[r(\beta_1 - \beta_2)\Omega(m) - c\beta_2]\,\delta_1(m) + c\beta_1\delta_2(m) - (\beta_1 - \beta_2)(c - r) = 0. \tag{14}$$

(ii) It holds that

$$L_1 = \left(\frac{\beta_2}{\beta_2 - \beta_1}\frac{c}{r}\right)^{\frac{1}{\beta_1}} \frac{K\beta_1}{\beta_1 - 1}.$$

(iii) For any $m \in (L_1, U(1))$ let $\tilde{U}_G(m)$ denote the unique positive solution of (14). $\tilde{U}_G : (L_1, U(1)) \to \mathbb{R}_+$ is continuous and strictly decreasing.

Equation (14) characterizes the entry boundary function $\tilde{U}_G(m)$ of Player 2 that would make m a part of the mixed strategy of Player 1. In other words, Lemma 5 means that

$$\begin{aligned} F(m,m) &< \Omega(m) \quad \text{if } \tilde{U}(m) < \tilde{U}_G(m), \\ F(m,m) &> \Omega(m) \quad \text{if } \tilde{U}(m) > \tilde{U}_G(m). \end{aligned} \tag{15}$$

Intuitively, if Player 2's strategy $\tilde{U}(m)$ is different than $\tilde{U}_G(m)$, then Player 1 is strictly better off with either of the pure strategy choices. If $\tilde{U}(m) < \tilde{U}_G(m)$ then Player 1 strictly prefers revealing at m than continuing. If $\tilde{U}(m) > \tilde{U}_G(m)$ then Player 1 prefers continuing at m.

It is important to realize that L_1 can be characterized based solely on the problem of Player 1 as the choice of L_1 is a nonstrategic decision. The equilibrium distribution G and in particular the level of L_0 must incorporate the strategic effects on the behavior of Player 2.

Let us denote the closure of the set where $\tilde{U}$ is not constant by $S(\tilde{U})$.

Lemma 6 (Common support) $\operatorname{supp}(G) = S(\tilde{U})$.

Lemma 6 means that $\tilde{U}$ is not constant only on the set that is in the support of G. This observation is used to prove the next lemma.

Lemma 7 (No gaps) *There are no gaps in* $\operatorname{supp}(G)$.

Finally we show that G has no atoms. Let us denote the probability that Player 1 stops exactly at stopping time corresponding to trigger strategy x by $J(x)$.

Lemma 8 (No atoms) *For $m \in [L_1, x_0)$, $J(m) = 0$.*

In summary, we have shown in this section that Player 1's strategy is a continuous distribution function G over lower triggers with no atoms (except possibly at x_0). On the support of G, Player 2's strategy $\tilde{U}(m)$ must be equal to $\tilde{U}_G(m)$, and this function is characterized by Lemma 5.

4.2 Learning and best response of the uninformed player

In this section we analyze the optimal strategy U of Player 2 given that Player 1 adopts an arbitrary continuous strategy G. As Player 1 randomizes over the lower trigger strategies, Player 2, while observing that the game is not stopped at a new minimum in the support of G, updates his belief about θ. Thus, provided that the signaling game is not stopped by time t, the belief π_t will depend on the running minimum M_t of X_t. The learning process is described by a function $\Pi(m)$ derived by Bayes rule

$$\Pi(m) = \frac{(1 - G(m))\,\pi_0}{1 - G(m)\pi_0},$$

such that $\pi_t = \Pi(M_t)$. In a similar fashion, the minimum process is used to update the distribution of the mixed strategy of the w type. Given a minimum m let $g(x;m) = G'(x)/\,(1 - G(m))$ be an updated density of G at $x \leq m$ conditional on $\theta = w$. For brevity we denote $g(m;m)$, the hazard function of G at m, by $g(m)$.

Player 2 chooses a stopping boundary $\tilde{U}$ in the state space $\left\{(x,m) \in \mathbb{R}^2_+ : x \geq m\right\}$ to maximize his expected discounted value taking into account, first, the possibility of learning the w type if the type is revealed at a random lower trigger and, second, the gradual learning about θ if new values in the support of G are reached. Denote this value by $V(x,m)$. The dependence on the running minimum places the problem of Player 2 in line with some work on (non-standard) lookback options (e.g. Guo and Shepp (2001)) and more general recent literature on optimal stopping of the maximum process (Peskir (1998)). Using the dynamic programming arguments and Itô's lemma, we derive that in the continuation region, for $x \in (m, \tilde{U}(m))$ with fixed m, $V(x,m)$ must satisfy the ordinary differential equation

$$rV(x,m) = \mu x V_x(x,m) + \frac{1}{2}\sigma^2 x^2 V_{xx}(x,m). \tag{16}$$

Note that, similar to differential equation (8) in the problem of the informed player, derivatives in m do not appear in equation (16). In the space (x,m), m changes only after hitting the diagonal $x = m$, and this property shall be employed below in the boundary condition (19).

The general solution to (16) is of the form

$$V(x,m) = A_1(m)x^{\beta_1} + A_2(m)x^{\beta_2}.$$

The coefficients $A_1(m)$ and $A_2(m)$ as well as the optimal boundary $\tilde{U}(m)$ are determined by considering extremes in the continuation region in (x,m). At the boundary $x = \tilde{U}(m)$ between the continuation and stopping region we require the familiar conditions of continuous and smooth fit, that is

$$V(\tilde{U}(m),m) = \Pi(m)\tilde{U}(m) - K, \tag{17}$$

$$V_x(\tilde{U}(m),m) = \Pi(m). \tag{18}$$

When $x = m$, that is on the diagonal in $\mathbb{R}^2_+$, the probability of facing the w type is $\Pi(m)$ and, upon a marginal change in m, the probability that the w type reveals is $-g(m)$. In Section 3.2 we derived that, if the w type reveals, Player 2 gets $W(m)$ given by (6). It follows that at $x = m$ it holds that[8]

$$V_m(m,m) = \Pi(m)g(m)\left(V(m,m) - W(m)\right). \tag{19}$$

To shorten notation let $\Delta_1(m) = (m/\tilde{U}(m))^{\beta_1} - (m/\tilde{U}(m))^{\beta_2}$ and $\Delta_2(m) = \beta_2(m/\tilde{U}(m))^{\beta_1} - \beta_1(m/\tilde{U}(m))^{\beta_2}$. Then after solving (16)-(19) we obtain that the best response strategy of Player 2 must satisfy the following differential equation

$$\begin{aligned}\tilde{U}'(m)[(1-\beta_1)(1-\beta_2)\Pi(m)\tilde{U}(m) - \beta_1\beta_2 K]\Delta_1(m) + \Pi'(m)\tilde{U}(m)^2(\Delta_1(m) - \Delta_2(m))\\ = -g(m)\Pi(m)\tilde{U}(m)[(\beta_1-\beta_2)W(m) - \Pi(m)\tilde{U}(m)(\Delta_1(m)-\Delta_2(m)) - K\Delta_2(m)]. \end{aligned}\tag{20}$$

The first term on the left-hand side stems from the dependence of the terminal payoff at the upper boundary on the minimum process and is standard in problems of optimal stopping of extremum processes (cf. equation (6) in Guo and Shepp (2001)). The second term on the left-hand side comes from the effect of learning from the minimum process. The term on the right-hand side captures the influence of the type revelation at a random lower trigger.

To identify a relevant boundary condition, we observe that when m reaches L_1, which is the lower bound on the support of G, the belief that $\theta = w$ is zero. At this point Player 2 never enters and thus the boundary condition at $m = L_1$ is

$$\tilde{U}(L_1) = U(0) = \infty.$$

4.3 Equilibrium

The analysis of both players' strategies of the previous sections provides ingredients for the equilibrium result stated in the proposition below. For technical reasons we shall formulate and prove the proposition for a 'perturbed' version of the game in which Player 2 gets εX_t if he contests the stake at time t from the strong type. In other words, the strong player is not a 'natural monopolist'. The fraction $\varepsilon > 0$ is assumed to be small (in the baseline model $\varepsilon = 0$). Our choice of the case $\varepsilon = 0$ so far stems from the attempt to simplify the exposition. On the other hand, this section shows that a small perturbation $\varepsilon > 0$ readily delivers the uniqueness of the equilibrium strategies.[9]

Assume therefore that $\varepsilon > 0$. The analysis of the previous sections can be accordingly adjusted without much difficulty. In particular, the lower bound on the support of G depends

[8] The boundary condition (19) closely corresponds with the normal reflection conditions in the standard optimal stopping problems of maximum (or minimum) process (see (13) and footnote 7); it reduces to $V_m(m,m) = 0$ if no event happens at m (that is, if either $\Pi(m)g(m) = 0$ or $V(m,m) = W(m)$).

[9] Similar types of assumptions to ensure equilibrium uniqueness have been used in other contexts. For example, in a war of attrition with incomplete information, Fudenberg and Tirole (1986) characterize the equilibrium strategies, as we do in our model, in terms of differential equations. The solution to the differential equation and thus the equilibrium are unique if the firms are not natural monopolists.

on ε and we denote it L_1^ε with $L_1^0 = L_1$. Then after denoting the strategies of Player 2 in the perturbed model by $\tilde{U}^\varepsilon$ and U^ε, we obtain $\tilde{U}^\varepsilon(L_1^\varepsilon) = U^\varepsilon(0) = \frac{1}{\varepsilon}K\beta_1/(\beta_1 - 1)$ (the derivation is similar to the one in Section 3.1). The boundedness of $U^\varepsilon(0)$ if $\varepsilon > 0$ ensures the uniqueness in the statement of Proposition 9. For simplicity, we suppress superscript ε from now on. The proposition does not include the trivial situation if $x_0 < L_1$, in which case the w type reveals with probability one at $t = 0$.

Proposition 9 *Let* $\hat{\Pi} = \Pi + (1-\Pi)\varepsilon$, $\varepsilon > 0$, *and*

$$f_1(m, G, \tilde{U}) = \frac{f_2(1-\pi_0 G)^2[(1-\beta_1)(1-\beta_2)\hat{\Pi}\tilde{U} - \beta_1\beta_2 K]\Delta_1}{(1-\varepsilon)\pi_0(1-\pi_0)\tilde{U}^2(\Delta_1-\Delta_2) - \pi_0(1-\pi_0 G)\tilde{U}[(\beta_1-\beta_2)W - \hat{\Pi}\tilde{U}(\Delta_1-\Delta_2) - K\Delta_2]},$$

$$f_2(m, G, \tilde{U}) = \frac{c\beta_2\tilde{U}\,(\delta_1-\delta_2)}{(\beta_1-\beta_2)\,(c\delta_2+r-c)\,m},$$

and denote by $(G^*, \tilde{U}^*)$ *the (unique) solution to the system of differential equations*

$$\begin{aligned} G'(m) &= f_1(m, G, \tilde{U}),\ G(L_1) = 1, && (21)\\ \tilde{U}'(m) &= f_2(m, G, \tilde{U}),\ \tilde{U}(L_1) = U(0) = \frac{\beta_1}{\beta_1-1}\frac{K}{\varepsilon}. && (22) \end{aligned}$$

Then a pair of strategies $(G, \tilde{U})$ *is the unique MPBE of the signaling game if*

$$G(m) = \begin{cases} 1 & \text{if } m < L_1, \\ G^*(m) & \text{if } L_1 \le m \le L_0, \\ 0 & \text{if } L_0 < m \le x_0,\ L_0 \neq x_0. \end{cases}$$

If $L_0 = x_0$, *then at* $t = 0$ *Player* 1 *of the* w *type reveals her type with probability* $G^*(L_0)$, *signals with probability* $1 - G^*(L_0)$ *and at* $t > 0$ *plays according to* G. *Player* 2 *contests at*

$$\tilde{U}(m) = \begin{cases} \tilde{U}^*(L_1) & \text{if } m < L_1, \\ \tilde{U}^*(m) & \text{if } L_1 \le m \le L_0, \\ \tilde{U}^*(L_0) & \text{if } L_0 < m \le x_0. \end{cases}$$

The lower bound on the support of G *is given by the solution* L_1 *of*

$$\left[(\beta_1-\beta_2)\,\Omega(L_1) - \beta_2\frac{c}{r}\right]\left(\frac{U(0)}{L_1}\right)^{\beta_1} + \beta_1\frac{c}{r}\left(\frac{U(0)}{L_1}\right)^{\beta_2} - (\beta_1-\beta_2)\left(\frac{c}{r}-1\right) = 0 \qquad (23)$$

and the upper bound is

$$L_0 = \min\{\bar{L}_0, x_0\},\ \bar{L}_0 = \inf\{m \ge L_1 : G^*(m) = 0\}.$$

Finally, $U(1) = K\beta_1/(\beta_1 - 1)$.

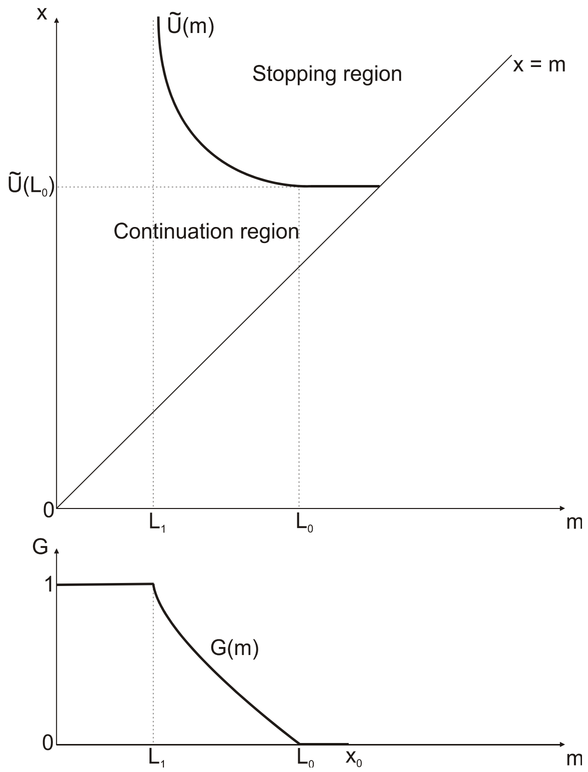

Figure 2: Equilibrium strategies.

A graphical representation of the equilibrium strategies is shown in Figure 2. The result can be interpreted as follows. For the outcome to satisfy the subgame perfectness criterion, Player 1 reveals at a random lower trigger with a continuous distribution G on some $[L_1, L_0]$. Player 2, while observing new minima reached in the support of G without the type revealed, updates his belief that he is facing the weak type (belief π decreases). Consequently, with decreasing m Player 2 requires a higher x to risk contesting the stake so $\tilde{U}(m)$ rises. Given that the game has not been stopped by the time x reaches L_1 (with $G(L_1) = 1$), the true type must be s and Player 2 contests at $U(0)$. The equilibrium slope of G in (21) is such that Player 2's best response to this G is $\tilde{U}$ (given by (22)) such that it is indeed optimal for Player 1 to choose a continuous G with support on $[L_1, L_0]$.

We would like to comment on the equilibrium in the limit as ε goes to zero. The problem is that the initial point $m = L_1$ becomes a singular point for the differential system (the system does not satisfy the Lipschitz continuity condition) and thus it has no unique solution. Such indeterminacy is not unusual in optimal stopping problems with extremum process. In a related setup with similar indeterminacy, Peskir (1998) introduced the maximality principle that determines the optimal trajectory of the optimal stopping boundary. Developing a corresponding principle in our strategic problem is beyond the scope of this paper. Nevertheless, building on the analogy to the maximality principle, we may anticipate that in our setup the equilibrium distribution $G^*(m)$ would be determined by choosing the maximal one among those solutions to (21) that are non-decreasing.

Our analysis of the two-type case can be adapted to the case of a continuum of types of the informed player. Such a setup would ease one technical issue, namely the informed player would not necessarily need to apply mixed strategies. On the other hand, with a continuum of types it might be more demanding to sustain the initial pooling. While in a binary signaling space, initial pooling strategy may be incentive compatible even with a continuum of types, typically we would not expect it in the case of a continuous signaling space (say in $\mathbb{R}$). The latter is the case in the application of the model to limit pricing in the next section, hence our focus on the two-type case; but in general the modelling will depend on the application in mind.

5 Application: Limit pricing

In this section we apply the dynamic stochastic signaling setup to a standard signaling situation known from industrial organization, namely limit pricing under incomplete information. Following Milgrom and Roberts (1982) we assume that the incumbent's cost is not directly observable by the potential entrant. When threatened by entry, the weak incumbent may, by setting low prices, pretend to be a strong one and thus discourage entry. Unlike the existing literature we study the dynamics of signaling and entry in a stochastic market.

The analysis here serves two purposes. First, it demonstrates how the general game of the preceding sections can be adapted to other functional forms of the players' payoffs. Specifically, the cost of signaling and payoffs of the informed player need to be functions of the driving diffusion process (here, the demand shock). Second, the model bridges the gap between the older non-game-theoretic literature on limit pricing, that was often dynamic and considered stochastic markets (Kamien and Schwartz (1971), Gaskins (1971) and Flaherty (1980)), and the game-theoretic equilibrium limit pricing that to a large degree invalidated the older explanations, but left us essentially with one-shot deterministic models.

5.1 Setup

We begin by describing the model setup. The incumbent firm, denoted by index 1, already operates in the market. Its profits depend on four factors. First, the profitability of the whole market evolves with a stochastic state variable Y following a geometric Brownian motion. Second, denoting the incumbent's cost type by $\theta \in \{w, s\}$, the incumbent's technology may be of low marginal cost C_1^s or high cost C_1^w per unit of time, $C_1^s < C_1^w$. Third, the incumbent may choose other than its monopoly price or quantity to imitate the behavior of another cost type. And lastly, profits depend on the presence of the entrant firm (firm 2). The entrant's marginal cost C_2 is known with certainty. We assume that upon entry the two firms compete in quantities in Cournot fashion. Below we show that these requirements on the profit flow function can be effectively captured by choosing an appropriate multiplicative constant for a stochastic state variable.

We assume the demand function is isoelastic and is subject to stochastic shocks. Specifically, the inverse demand function of total output Q at time t is given by

$$P_t(Q) = Y_t Q^{-\frac{1}{\gamma}}.$$

$Y = \{Y_t : t \geq 0\}$ is a stochastic state variable following a geometric Brownian motion with drift μ_Y, volatility σ_Y and a standard Brownian motion Z. γ is the demand elasticity and we assume that $\gamma > 1$.

It is straight-forward to derive that the profit flow of the unconstrained (that is, not facing a potential entrant) monopolist of type θ at each state Y_t is

$$\frac{Y_t^\gamma}{\gamma}\left(\frac{\gamma C_1^\theta}{\gamma - 1}\right)^{1-\gamma}. \tag{24}$$

Define now a new variable $X_t = f(Y_t) = (Y_t)^\gamma$. By Itô's lemma we have

$$dX_t = f' dY_t + \frac{1}{2} f'' dY_t^2 = \mu X_t dt + \sigma X_t dZ_t,$$

where $\mu = \gamma\mu_Y + \frac{1}{2}\gamma(\gamma - 1)\sigma_Y^2$ and $\sigma = \gamma\sigma_Y$ are constants, and f' and f'' denote the first and second order derivatives. Therefore, X is also a geometric Brownian motion adapted to the same filtration.

Note that $\frac{1}{\gamma}\left(\frac{\gamma C_1^\theta}{\gamma-1}\right)^{1-\gamma}$ in (24) is constant over time, thus the profit flow in (24) may be expressed as a constant times a geometric Brownian motion. A similar equivalence can be shown for profit flows under duopolistic competition and in case the monopolist chooses quantities corresponding to optimal quantities of another cost type (i.e. imitates optimal behavior of another type). In particular, the profit flow of the incumbent of type θ is a product of the market state variable X and a constant equal to either of

$$M^\theta = \frac{1}{\gamma}\left(\frac{\gamma C_1^\theta}{\gamma - 1}\right)^{1-\gamma}, \tag{25}$$

$$M^\theta(\tilde{C}) = \frac{\gamma\tilde{C} - (\gamma - 1)C_1^\theta}{\gamma - 1}\left(\frac{\gamma\tilde{C}}{\gamma - 1}\right)^{-\gamma}, \tag{26}$$

$$D_1^\theta = \begin{cases} \frac{\left[\gamma C_2 - (\gamma-1)C_1^\theta\right]^2}{(2\gamma-1)\left(C_1^\theta + C_2\right)}\left[\frac{\gamma\left(C_1^\theta + C_2\right)}{2\gamma - 1}\right]^{-\gamma} & \text{if } C_1^\theta < \frac{\gamma}{\gamma-1}C_2, \\ 0 & \text{otherwise,} \end{cases} \tag{27}$$

depending on whether the incumbent is a monopolist choosing its unconstrained monopoly strategy (M^θ), or if the incumbent is a monopolist imitating the monopoly strategy of a firm with marginal cost $\tilde{C}$ ($M^\theta(\tilde{C})$), or if the incumbent firm operates in a duopoly (D_1^θ). The relationships between the profit constants are unsurprising, namely $M^\theta > D_1^\theta \geq 0$, $M^s > M^w$, $M^\theta > M^\theta(\tilde{C})$ and $D_1^s > D_1^w$ for all θ and all $\tilde{C} \neq C^\theta$. From (26) it follows that the θ-type incumbent imitating the pricing strategy of a monopolist with cost $\tilde{C}$ has negative profits if $\tilde{C} < \frac{\gamma-1}{\gamma}C_1^\theta$. Equation (27) says that the θ-type incumbent is out of the market after entry if the entrant's cost is less than $\frac{\gamma-1}{\gamma}C_1^\theta$.

The incumbent's type is known to the incumbent firm itself but not, at the initial point of time, to the potential entrant. Instead, the prior probability that $\theta = w$ is π_0 and is known to the entrant. Upon entry, firm 2 pays the entry cost of K and learns the cost type of the incumbent. After the entrant has entered the market, its profits are affected by the cost level of the incumbent. Given that firm 1 is of the θ type, firm 2's profit flow after entry can be expressed as the product of X and a constant, where the constant is

$$D_2^\theta = \begin{cases} \frac{\left[\gamma C_1^\theta-(\gamma-1)C_2\right]^2}{(2\gamma-1)\left(C_1^\theta+C_2\right)}\left[\frac{\gamma\left(C_1^\theta+C_2\right)}{2\gamma-1}\right]^{-\gamma} & \text{if } C_2 < \frac{\gamma}{\gamma-1}C_1^\theta, \\ 0 & \text{otherwise.} \end{cases} \tag{28}$$

The lower the incumbent's cost the less profitable is entry, that is $D_2^w > D_2^s$. If the entrant knows the incumbent's type to be θ and $C_2 \geq \frac{\gamma}{\gamma-1}C_1^\theta$, then the entrant cannot make positive profits and thus never enters.

We make the following assumptions about the cost structure. (1) There is a profitable entry in the market against the w type: $D_2^w > 0$. (2) The incumbent of type w prefers imitating the s-type than facing immediate entry: $M^w(C_1^s) \geq D_1^w$. (3) The incumbent of type s has its marginal cost slightly higher than that of natural monopolist: $C_1^s = (1+\bar{\varepsilon})\frac{\gamma-1}{\gamma}C_2$, where $\bar{\varepsilon}$ is assumed to be a small positive number. Assumption (1) ensures that the game is interesting. Assumption (2) is a necessary condition for the incumbent of type w to engage in signaling (it corresponds the assumption that $c < r$ in the generic game). By assumption (3) we concentrate our attention on the case corresponding to the analysis in Sections 2-4. The strong incumbent is strong enough to have no incentives to signal its type, yet the entrant gets a small share of the market after entry to ensure uniqueness of the equilibrium strategies (analogously to the ε-perturbation in Section 4.3). Assumption (3) can be rewritten in terms of entrant's profits as $D_2^s = \varepsilon D_2^w$ for some (small) ε corresponding to $\bar{\varepsilon}$.

5.2 Strategies and equilibrium

The game has essentially the same structure as the general stochastic signaling game studied in this paper. The games differ in the flows and payoffs available for the two players. The description of the strategies and equilibrium concept carries over from Section 2.2. The entrant, given its belief π about the incumbent's type, chooses a strategy to enter at a sufficiently large market, that is whenever $x \geq U(\pi)$. The weak incumbent's strategy is to stop charging limit prices if the market variable x falls below $L(\pi)$. The second dimension of the incumbent's strategy is the choice of prices, in particular we need to specify the limit prices, i.e. the prices set in the pooling outcome. As in the simple one-shot games, in principle, a continuum of prices may be sustained in a pooling equilibrium. While we do not develop formal refinement criteria for the continuous-time game, we focus on the most plausible outcome, that is sustained by the least cost criterion. As a consequence, we analyze efficient pooling at the prices of the strong incumbent. Let us denote the profit flow coefficient of the w-type charging the monopoly price of the s-type by M^p, where $M^p = M^w(C_1^s)$.

The payoffs of the firms are in flows of profit and a fixed cost of entry. After entry, the weak incumbent faces a profit decline by factor $M^w - D_1^w$. The per period cost rate of signaling is now the difference between the profit flow with limit pricing and the unconstrained monopoly profit, that is $M^w - M^p$. If the entrant enters the market with belief π, it pays the entry cost K and its expected profit flow coefficient is $\hat{\pi} D_2^w$ where $\hat{\pi} = \pi + (1-\pi)\varepsilon$. Putting these elements together we write the total expected payoff functions for both firms given a pair of strategies (L, U) and starting values (x, π) as follows:

$$
\begin{aligned}
R_1(x,\pi;L,U) &= E_{x,\pi}\left[\int_0^{\tau_L \wedge \tau_U} e^{-rt} M^p X_t dt + \int_{\tau_U}^{\infty} e^{-rt} D_1^w X_t dt 1_{\tau_L \geq \tau_U}\right. \\
&\quad \left. + \left(\int_{\tau_L}^{\tau_w} e^{-rt} M^w X_t dt + \int_{\tau_w}^{\infty} e^{-rt} D_1^w X_t dt\right) 1_{\tau_L < \tau_U}\right], \\
R_2(x,\pi;L,U) &= E_{x,\pi}\left[\left(\hat{\pi}_{\tau_U}\int_{\tau_U}^{\infty} e^{-rt} D_2^w X_t dt - K\right) 1_{\tau_L \geq \tau_U} + \left(\int_{\tau_w}^{\infty} e^{-rt} D_2^w X_t dt - K\right) 1_{\tau_L < \tau_U}\right].
\end{aligned}
$$

The stopping times are defined analogously to Section 2.2. The discussion of Section 4.1 applies and in equilibrium the incumbent plays a mixed strategy, which is a continuous distribution function G over trigger strategies.

To develop the notation used in the equilibrium proposition below, let us now consider the terminal payoffs after the incumbent's type is revealed. The entrant's value, given that $\pi = 1$, is

$$W(x) = \left(\frac{x}{U(1)}\right)^{\beta_1} \left(d_2 U(1) - K\right),$$

where $d_2 = D_2^w/(r-\mu)$. The optimal entry trigger is

$$U(1) = \frac{\beta_1}{1-\beta_1}\frac{K}{d_2}.$$

Given the entrant's strategy $U(1)$, the value of the incumbent of the w type is

$$\Omega(x) = \frac{M^w x}{r-\mu} - \left(\frac{x}{U(1)}\right)^{\beta_1} \frac{(M^w - D_1^w)U(1)}{r-\mu}.$$

Detailed derivations, that are similar to those in the baseline signaling game, and the proof of the equilibrium result in the following proposition are relegated to Appendix.

Proposition 10 *Let* $\hat{\Pi} = \Pi + (1-\Pi)\varepsilon$ *and*

$$f_1(m, G, \tilde{U}) =$$
$$\frac{f_2(1-\pi_0 G)^2[(1-\beta_1)(1-\beta_2)\hat{\Pi} d_2 \tilde{U} - \beta_1\beta_2 K]\Delta_1}{(1-\varepsilon)\pi_0(1-\pi_0)d_2\tilde{U}^2(\Delta_1-\Delta_2) - \pi_0(1-\pi_0 G)\tilde{U}[(\beta_1-\beta_2)W - \hat{\Pi} d_2\tilde{U}(\Delta_1-\Delta_2) - K\Delta_2]},$$

$$f_2(m,G,\tilde{U}) = \frac{(\beta_2-1)(M^w-M^p)m\,(\delta_1-\delta_2)}{(\beta_1-\beta_2)\left[(M^w-M^p)\tilde{U}\delta_2+(M^p-D_1^w)m\right]},$$

and denote by $(G^*,\tilde{U}^*)$ *the (unique) solution to the system of differential equations*

$$G'(m) = f_1(m,G,\tilde{U}),\ G(L_1)=1, \tag{29}$$

$$\tilde{U}'(m) = f_2(m,G,\tilde{U}),\ \tilde{U}(L_1)=U(0)=\frac{\beta_1}{\beta_1-1}\frac{K}{\varepsilon d_2}. \tag{30}$$

Then a pair of strategies $(G,\tilde{U})$ *is the unique MPBE of the signaling game if*

$$G(m)=\begin{cases} 1 & if\ m<L_1, \\ G^*(m) & if\ L_1\le m\le L_0, \\ 0 & if\ L_0<m\le x_0,\ L_0\neq x_0. \end{cases}$$

If $L_0=x_0$*, then at* $t=0$ *the incumbent of the* w *type reveals its type with probability* $G^*(L_0)$*, signals with probability* $1-G^*(L_0)$*, while at* $t>0$ *it plays according to* G*. The entrant contests at*

$$\tilde{U}(m)=\begin{cases} \tilde{U}^*(L_1) & if\ m<L_1, \\ \tilde{U}^*(m) & if\ L_1\le m\le L_0, \\ \tilde{U}^*(L_0) & if\ L_0<m\le x_0. \end{cases}$$

The lower bound on the support of G *is given by the solution* L_1 *of*

$$\{(\beta_1-\beta_2)(r-\mu)\Omega(L_1)-[(\beta_1-1)M^w+(1-\beta_2)M^p]\,m\}\left(\frac{U(0)}{L_1}\right)^{\beta_1} + (\beta_1-1)(M^w-M^p)\,m\left(\frac{U(0)}{L_1}\right)^{\beta_2}+(\beta_1-\beta_2)(M^w-M^p)\,\tilde{U}(L_1)=0 \tag{31}$$

and the upper bound is

$$L_0=\min\{\bar{L}_0,x_0\},\ \bar{L}_0=\inf\{m\ge L_1 : G^*(m)=0\}.$$

Finally, $U(1)=\frac{1}{d_2}K\beta_1/(\beta_1-1)$.

5.3 Implications

By introducing continuous time dynamics and uncertainty, we can derive some interesting implications that are unavailable in the existing game theoretic models of limit pricing. We formulate here several observations that are direct consequences of the equilibrium result and are of interest as either empirical predictions or policy recommendations.

Observation 1 (Price dynamics) When the incumbent reveals its type at a random lower trigger, prices increase in a decreasing market.

Under our assumption of isoelastic demand and constant marginal cost, (unconstrained) monopoly prices are constant. Yet the price dynamics under limit pricing may take an unusual pattern with prices increasing in a decreasing market. (Under other demand-cost specifications, this would translate into an unexpected price increase in a decreasing market.) First of all, this observation provides an empirical prediction that could be confronted with the data. Second, we provide a potential policy implication of this remarkable price dynamics. Based on one-shot models, limit (or predatory) pricing can be detected by comparing prices to marginal costs. This is the usual approach of antitrust authorities (see e.g. Sufrin and Jones (2004)). However, marginal costs are in general difficult to observe and thus in the regulatory practice undesirable limit pricing is difficult to discover and prove. The task to prove limit pricing practices is even more daunting when taking into account the asymmetric-information arguments, with the assumption that costs are unobservable by competitors. The advantage of our dynamic model is that it implies that the easily observable price dynamics may reveal limit pricing practices of incumbents. In particular, increasing prices in a decreasing market indicate that the incumbent has used prices to deter entrants.

Observation 2 (Path dependence) The entrant's decision to enter depends on historical demand.

Our model shows that market dynamics (that is in our setup the transition from monopoly to duopoly) exhibits path dependence in that the entrant's decision to enter depends on historical demand. This is despite the fact that the demand shocks are Markovian and the current demand level is a sufficient statistic for the future distribution. Yet, because a market downturn in the past made it the more likely that the weaker type of incumbent would have stopped using limit pricing, the probability of facing the strong incumbent increases under the limit pricing regime. In other words, a demand slump polarizes entry timing, the entry happens either early against the weak incumbent or late against an uncertain type.

Observation 3 (Learning and entry) The learning effect postpones entry.

Under complete information, the ratio of expected discounted profits at entry to the fixed cost is $\beta_1/(\beta_1 - 1)$ in both cases if $\pi = 0$ and $\pi = 1$ (recall that $\beta_1 > 1$, and the reason that the ratio exceeds 1 is that it incorporates the value of waiting, the standard result from the theory of investment under uncertainty, see, e.g., Dixit and Pindyck (1994)). Yet, when there is still incomplete information about the incumbent type, that is if $\pi \in (0, 1)$, the same ratio, that is $\hat{\pi} d_2 U(\pi)/K$, is larger than $\beta_1/(\beta_1 - 1)$. The difference stems from the learning effect. Whenever $\pi \in (0, 1)$, the entrant takes into account that over time it may learn more about the incumbent's cost type realization and make a more knowledgeable decision in the future. Consequently, the entrant postpones the entry decision and requires higher expected profits to enter.

6 Conclusions

We have presented a model of dynamic signaling in a stochastic environment and showed that such a setup brings novel strategic interactions between the informed and uninformed players. In our setting the payoffs (stake in the game) depend on the type of the informed player and follow a diffusion process. For a given belief about the type of the informed player, the uninformed player has incentives to stop the signaling game and contest the potential payoff at a sufficiently high stake. On the other hand, the informed player has incentives to stop signaling at a sufficiently low stake. We characterize a Markov equilibrium in which the two players choose threshold strategies on the stake to stop the signaling game. Interestingly, the minimum process of the stake in the game captures the Bayesian learning of the uninformed player. Based on this observation, we could use the techniques of optimal stopping of extremum processes. The dynamic stochastic environment causes the gradual evolution from pooling via semi-pooling to separating outcome.

We apply the model to analyze dynamic limit pricing under uncertain demand. The dependence on the minimum process drives the path-dependence of the outcome of the game. In the limit pricing application, this feature brings a path-dependent market structure. The model also implies that observable price dynamics may reveal entry-deterring pricing practices. We find that a dynamic environment with incomplete information creates pre-entry learning opportunities about the incumbent's type and so causes delayed entry.

The model may be of interest in other applications and particularly valuable in corporate finance. In corporate finance theory, both asymmetric information and continuous-time dynamics driven by diffusion processes play prominent roles. Our paper merges these two, so far independent, modeling environments.

A Appendix: Proofs

Proof of Proposition 1. For any $\mu \neq 0$, using that the solution to the differential equation for X_t if $\sigma = 0$ is $X_t = x_0 e^{\mu t}$, we derive the discount factor

$$e^{-rt} = \left(\frac{x_0}{X_t}\right)^{\frac{r}{\mu}}. \tag{32}$$

Suppose now that $\mu < 0$. Because x decreases deterministically, Player 2 enters whenever he breaks even in expectations. To see it, first note that any threat of Player 2 to enter earlier to induce type revelation of the w type is an empty threat. The w type would rather not reveal and make Player 2 believe that she is the s type. Second, for any given π there is no value in waiting as x decreases in time. So Player 2 enters whenever $X_t \geq U(\pi_t) = K/\pi_t$. If Player 1 reveals at some trigger above K, it spurs an immediate entry. Clearly, Player 1 obtains the highest payoff from signaling if she signals from $t = 0$ until the time when x reaches K and thus prevents any entry. The incentive compatibility constraint for such a signaling pattern is satisfied at $t = 0$ whenever

$$-\frac{c}{r}\left[1-\left(\frac{x_0}{K}\right)^{\frac{r}{\mu}}\right] > -1. \tag{33}$$

The left-hand side represents the cost of signaling when x is between x_0 and K (using (32)), and the right-hand side represents the payoff if the type is revealed. It is easy to verify that if the constraint (33) holds at x_0, it remains binding for any $x \in (K, x_0)$. So if $\Gamma_1(x_0) > 0$, Player 1 would only reveal whenever $X_t \le L = K$. If $\Gamma_1(x_0) > 0$, the incentive compatibility constraint does not hold, and Player 1 does not signal at all, so $L = x_0$. Finally, when either $\Gamma_1(x_0) > 0$ or $\Gamma_1(x_0) < 0$, there does not exist an equilibrium where the w type randomizes between revealing and signaling as she strictly prefers one of the alternatives. ■

Proof of Proposition 2. Player 2 cannot use a threat to enter early to induce type revelation of the w type. The w type would rather not reveal and make Player 2 believe that she is the s type. So for any π Player 2 chooses the level of the entry trigger in x to maximize its expected payoff $(x_0/x)^{r/\mu}(\pi x - K)$ (using (32)). This yields that Player 2 enters as soon as $X_t \ge U^d(\pi) = (K/\pi)\, r/(r-\mu)$.

Player 1 of the w type gains nothing from signaling if she reveals at $x \in (x_0, U(1)]$ and faces an immediate entry if she reveals at $x \in (U(1), U(\pi_0))$. It follows that, if Player 1 decides for signaling, the most profitable signaling strategy is to signal until Player 2 enters at $U(\pi_0)$. The incentive compatibility constraint for such a signaling pattern is satisfied at $t = 0$ whenever

$$-\frac{c}{r} - \left(\frac{x_0}{U(\pi_0)}\right)^{\frac{r}{\mu}}\left(1-\frac{c}{r}\right) > -\left(\frac{x}{U(1)}\right)^{\frac{r}{\mu}}. \tag{34}$$

The left-hand side represents the cost of signaling when x is between x_0 and $U(\pi_0)$ (using (32)), and the right-hand side represents the payoff if the type is revealed.

We now show that if the incentive compatibility constraint (34) holds at x_0 it will hold at any $x \in (x_0, U(\pi_0))$. At any $x \in [x_0, U(1)]$ the condition for signaling equivalent to (34) can be written as

$$\begin{aligned} & -\frac{c}{r} + \left(\frac{x}{U(\pi_0)}\right)^{\frac{r}{\mu}}\left(-1+\frac{c}{r}\right) + \left(\frac{x}{U(1)}\right)^{\frac{r}{\mu}} \\ \ge\ & \left(\frac{x}{x_0}\right)^{\frac{r}{\mu}}\left[-\frac{c}{r} + \left(\frac{x_0}{U(\pi_0)}\right)^{\frac{r}{\mu}}\left(-1+\frac{c}{r}\right) + \left(\frac{x_0}{U(1)}\right)^{\frac{r}{\mu}}\right] \ge 0, \end{aligned}$$

where in the last inequality we use (34). Similar inequalities can be written for $x \in (U(1), U(\pi_0)]$. This proves that if (34) holds then Player 1 would never reveal until entry. So if $\Gamma_2(x_0) > 0$, Player 1 would never reveal, or, in terms of a trigger strategy, reveal whenever $X_t \le L = 0$ (precisely, at any other inaccessible value in $(0, x_0)$). If $\Gamma_2(x_0) < 0$, pure strategy pooling is not an equilibrium. The (pure strategy) separating outcome is an equilibrium, if the weak type does not want to imitate the strong type, that is if

$$-\frac{c}{r} \leq -\left(\frac{x}{U(1)}\right)^{\frac{r}{\mu}}. \tag{35}$$

The left-hand side represents the cost of signaling when Player 2 believes the type is s and never enters. It follows that if $\Gamma_3(x_0) \leq 0$, the unique equilibrium is a pure strategy of player to reveal at x_0. If neither (34) nor (35) holds, then the w type cannot put probability one on either pure strategy. The probabilities p and $(1-p)$ follow from Bayes rule and the indifference of Player 1 for signaling and revealing at $t = 0$. ■

Proof of Lemma 5. (i) Combining conditions (11) and (12) with the general solution (9) we obtain

$$\begin{aligned} B_1(m) &= \frac{1}{\beta_1 - \beta_2}\left[(\beta_1 - \beta_2)\,\Omega(m) - \frac{c}{r}\beta_2\right] m^{-\beta_1}, \\ B_2(m) &= \frac{c}{r}\frac{\beta_1}{\beta_1 - \beta_2} m^{-\beta_2}, \end{aligned}$$

for all $m \in \text{supp}(G)$. We note that after substituting $B_1(m)$ and $B_2(m)$ into (13), (13) holds as an identity. Then substituting $B_1(m)$ and $B_2(m)$ in (10) yields (14).

(ii) In addition to $F(L_1, L_1) = \Omega(L_1)$ and $F_x(L_1, L_1) = \Omega'(L_1)$, the value must be bounded as x goes to infinity. Applying all three conditions to the general solution of (8) yields part (ii).

(iii) Clearly, it must hold that $\tilde{U}(m) \geq m$. Then the derivative of the left hand side of (14) with respect to $\tilde{U}$ is

$$\left[r\beta_1\left(\beta_1 - \beta_2\right)\delta_1(m)\Omega(m) - c\beta_1\beta_2\left(\delta_1(m) - \delta_2(m)\right)\right]\tilde{U}(m)^{-1} < c\beta_1\beta_2\delta_2(m)\tilde{U}(m)^{-1} < 0,$$

where the first inequality follows from the fact that $\Omega(m) < -\frac{\beta_2}{\beta_2 - \beta_1}\frac{c}{r}$ for $m > L_1$ (using part (ii) of the lemma). Moreover, observing that the left hand side of (14) is continuous in positive $\tilde{U}$ and it diverges to $+\infty$ at $\tilde{U} = 0$ and diverges to $-\infty$ as $\tilde{U}$ goes to infinity, we conclude that there is a unique positive root $\tilde{U}_G$. As the left hand side of (14) is strictly positive for $\tilde{U}(m) = m$ (with $m \in (L_1, U(1))$), it follows that $\tilde{U}_G(m) > m$, as expected. Finally, a straightforward application of the implicit function theorem and some algebra delivers that $\tilde{U}_G(\cdot)$ is strictly decreasing. ■

Proof of Lemma 6. Suppose that $m \in \text{supp}\,(G)$ and $m \notin S(\tilde{U})$. By Lemma 5, $m \in \text{supp}\,(G)$ implies that $\tilde{U}(m) = \tilde{U}_G(m) < \infty$. $m \notin S(\tilde{U})$ means that there is an open neighborhood of m, $b(m)$, such that for all $m' \in b(m)$, $\tilde{U}(m') = \tilde{U}(m)$. Then at any $m' + \epsilon < m$, with some $\epsilon > 0$ and $m' \in b(m)$, $\tilde{U}(m' + \epsilon) > \tilde{U}_G(m' + \epsilon)$ as $\tilde{U}_G$ is strictly decreasing by Lemma 5(iii). But then, by (15), Player 1 strictly prefers revealing than continuing at $m' + \epsilon$. So, if no revelation is observed at $m' + \epsilon$, then Player 1 must be of s type and $\tilde{U}(m') = \infty \neq \tilde{U}(m)$. This contradicts that $\tilde{U}(m') = \tilde{U}(m)$ and consequently it can not be that $m \in \text{supp}\,(G)$ and $m \notin S(\tilde{U})$.

Next suppose that $m \notin \text{supp}\,(G)$ and $m \in S(\tilde{U})$. It means that there is an open neighborhood of m, $b(m)$, such that for all $m' \in b(m)$, $G(m') = G(m)$. This means also that $m' \in b(m)$ the belief π cannot be different as there is no information revealed. As for all $m' \in b(m)$ the belief is the same and the probability of reaching minima in the support of G are the same, it can not be optimal to play $\tilde{U}(m') \neq \tilde{U}(m)$. Thus $b(m) \nsubseteq S(\tilde{U})$ contradicting that $m \in S(\tilde{U})$. ■

Proof of Lemma 7. Suppose that there is a gap (a, b) over which G is constant and a and b belong to the support of G (recall that supp(G) is a closed set). Then, by Lemma 5, $\tilde{U}(a) = \tilde{U}_G(a)$ and $\tilde{U}(b) = \tilde{U}_G(b)$. By Lemma 6, $\tilde{U}$ is constant over (a, b). If Player 1 does not put any positive probability on strategies in (a, b), then for any $m \in (a, b)$, it must hold by (15) that $\tilde{U}(m) \geq \tilde{U}_G(m)$. Then, as $\tilde{U}$ is constant in (a, b) and $\tilde{U}_G$ is strictly decreasing (Lemma 5(iii)), it follows that $\tilde{U}(m) \geq \tilde{U}(a) > \tilde{U}_G(m)$ for all $m \in (a, b)$. Thus, by (15), the continuation payoff $F(m, m)$ is strictly larger than terminal payoff $\Omega(m)$ for all $m \in (a, b)$. We also note that $\Omega(\cdot)$ is a continuous function. Then the following inequality holds

$$F(b, b) = \Omega(b) = \lim_{m \uparrow b} \Omega(m) < \lim_{m \uparrow b} F(m, m).$$

But then b cannot be in G as by an infinitesimal deviation and not revealing at b, Player 1 gets a benefit that is bounded away from zero. Consequently there cannot be gaps in supp (G). ■

Proof of Lemma 8. Suppose $J(l) = p > 0$ for some $l \in (0, x_0)$. As $l \in$ supp (G), $F(l, l) = \Omega(l)$ and $\tilde{U}(l) = \tilde{U}_G(l)$ by Lemma 5. Suppose that at the time l is reached for the first time, the belief is $\pi \in (0, 1)$. If Player 1 does not stop, Player 2 uses Bayes rule to update his belief to $\pi' = \frac{(1-p)\pi}{1-\pi p} < \pi$. Let us denote $\lim_{m \uparrow l} \tilde{U}(m)$ by $\tilde{U}^-(l)$ and $\lim_{m \uparrow l} \tilde{U}_G(m)$ by $\tilde{U}_G^-(l)$. We have two cases to consider.

Case 1. Suppose that $U(\pi') > U(\pi)$, that is $\tilde{U}^-(l) > \tilde{U}(l)$. By Lemma 5(iii), U_G is continuous, so $\tilde{U}^-(l) > \tilde{U}_G^-(l)$. But then by the same argument as in Lemma 7, Player 1 faces a jump in the value and by an infinitesimal deviation gets a benefit that is bounded away from zero. Formally, the following inequality holds:

$$F(l, l) = \Omega(l) = \lim_{m \uparrow l} \Omega(m) < \lim_{m \uparrow l} F(m, m).$$

Hence l is not played in the mixed strategy G and $J(l) = 0$.

Case 2. Suppose that $U(\pi') \leq U(\pi)$, that is $\tilde{U}^-(l) \leq \tilde{U}(l)$. We shall show that Player 2's best response is never $U(\pi') \leq U(\pi)$ if $p > 0$. To do this let us consider Player 2's best response problem given G and $J(l) = p > 0$. Let $V(x, l)$ be the value of Player 2 in this best response problem. In the continuation region, that is for $x \in (l, \tilde{U}(l))$, $V(x, l)$ must satisfy the Bellman-type equation

$$rV(x, l) = \mu x V_x(x, l) + \frac{1}{2}\sigma^2 x^2 V_{xx}(x, l).$$

At the boundaries the following conditions hold

$$\begin{aligned} V(\tilde{U}(l),l) &= \pi\tilde{U}(l)-K, \\ V_x(\tilde{U}(l),l) &= \pi, \\ V(l,l) &= \pi p W(l)+(1-\pi p)V^-(l,l), \end{aligned} \tag{36}$$

where $V^-(l,l)=\lim_{m\uparrow l}V(m,m)$ denotes the continuation value just after the minimum at l is reached. Let $\Delta_1=(l/\tilde{U}(l))^{\beta_1}-(l/\tilde{U}(l))^{\beta_2}$ and $\Delta_2=\beta_2(l/\tilde{U}(l))^{\beta_1}-\beta_1(l/\tilde{U}(l))^{\beta_2}$. Then using the general solution to the differential equation and the two first boundary conditions we obtain that

$$V(l,l)=\frac{1}{\beta_1-\beta_2}\left[\pi\tilde{U}(l)(\Delta_1-\Delta_2)+K\Delta_2\right]. \tag{37}$$

$V^-(x,l)$ must satisfy the same differential equation, but the boundary conditions at $\tilde{U}^-(l)$ become

$$\begin{aligned} V^-(\tilde{U}^-(l),l) &= \pi'\tilde{U}^-(l)-K, \\ V_x^-(\tilde{U}^-(l),l) &= \pi'. \end{aligned}$$

Solving for $V^-(x,l)$ we find

$$V^-(l,l)=\frac{1}{\beta_1-\beta_2}\left[\pi'\tilde{U}^-(l)(\Delta_1^- -\Delta_2^-)+K\Delta_2^-\right],$$

where $\Delta_1^-=(l/\tilde{U}^-(l))^{\beta_1}-(l/\tilde{U}^-(l))^{\beta_2}$ and $\Delta_2^-=\beta_2(l/\tilde{U}^-(l))^{\beta_1}-\beta_1(l/\tilde{U}^-(l))^{\beta_2}$. As $V^-(l,l)$ increases in $\tilde{U}$ (as long as $\pi'\tilde{U}^-(l)\geq\frac{\beta_2}{\beta_2-1}U(1)$, but this must be the case in the best response of Player 2, as the point with equality is where $V^-(l,l)$), we can write the following inequality

$$V^-(l,l)\leq\frac{1}{\beta_1-\beta_2}\left[\pi'\tilde{U}(l)(\Delta_1^- -\Delta_2^-)+K\Delta_2^-\right].$$

Substituting this inequality together with (37) and $\pi'=\frac{(1-p)\pi}{1-\pi p}$ in (36) yields

$$p\frac{1}{\beta_1-\beta_2}\left[\tilde{U}(l)(\Delta_1-\Delta_2)+K\Delta_2\right]\leq pW(l). \tag{38}$$

The term on the left hand side equals $pW(l)$ if $\tilde{U}(l)=U(1)$ and increases in $\tilde{U}(l)$ if $\tilde{U}(l)>U(1)$. As $l\in\operatorname{supp}(G)$ then, by Lemma 5, $\tilde{U}(l)>U(1)$, hence the weak inequality (38) holds only if $p=0$.

Note that the argument in Case 1 does not hold for the initial state (x_0,x_0) at $t=0$. At (x_0,x_0) it can be that $F(x_0,x_0)<\Omega(x_0)$. In this case Player 1 prefers revealing above continuing and reveals with probability 1 if $x_0\leq L_1$ or randomizes between continuing and revealing if $x_0>L_1$. ■

Proof of Proposition 9. In the proof we refer to the results in Sections 3 and 4.1-4.2

stated there for the limit case $\varepsilon = 0$, while pointing to the necessary adjustments as $\varepsilon > 0$. If $x_0 > L_1$, then by Lemma 4 there exist no equilibrium in pure strategies. By Lemmas 7 and 8, the mixed strategy G of Player 1 has to be a continuous function with a support on some $[L_1, L_0]$. L_1 is given in Lemma 5(ii) in the limit case $\varepsilon = 0$. If $\varepsilon > 0$, a similar derivation, but with an additional boundary condition

$$F(U(0), L_1) = -1,$$

yields the implicit equation for L_1 as stated in the proposition. From Lemma 5, $m \in (L_1, x_0)$ is in the support of G only if $\tilde{U}(m)$ satisfies (14), which after differentiation gives $\tilde{U}'(m) = f_2(m, G, \tilde{U})$, with initial value condition $\tilde{U}(L_1) = \frac{1}{\varepsilon} K\beta_1/(\beta_1 - 1)$.

From Section 4.2 it follows that Player 2 chooses the given $\tilde{U}(m)$ if $G(m)$ satisfies equation (20). If $\varepsilon > 0$, the boundary conditions (17) and (18) are substituted with

$$\begin{aligned} V(\tilde{U}(m), m) &= \hat{\Pi}(m)\tilde{U}(m) - K, \\ V_x(\tilde{U}(m), m) &= \hat{\Pi}(m). \end{aligned}$$

Combination of these and (19) with the solution to (16) and some reorganization yield $G'(m) = f_1(m, G, \tilde{U})$. The initial value condition $G(L_1) = 1$ follows from the construction of L_1.

The upper bound on the support of G is then $L_0 = \min\{\bar{L}_0, x_0\}$, with $\bar{L}_0 = \inf\{m \geq L_1 : G^*(m) = 0\}$. If $L_0 < x_0$, then clearly $G(m) = 0$ and $\tilde{U}(m) = \tilde{U}(L_0)$ for $m > L_0$. If $\bar{L}_0 < x_0$, then neither continuing nor stopping is a pure strategy equilibrium at $t = 0$ (with the arguments parallel to those in Lemma 4). In the mixed strategy at $t = 0$, Player 1 randomizes to be indifferent between revealing and signaling and chooses probabilities $G^*(L_0)$ and $1 - G^*(L_0)$, respectively.

The complete information threshold $U(1)$ is derived in Section 3.1.

Finally, we confirm the uniqueness of the equilibrium. It is not difficult to verify that the initial value problem (21)-(22) satisfies the Lipschitz condition as long as $\varepsilon > 0$. Thus there is a unique solution $(G^*, \tilde{U}^*)$ if the informed player applies a continuous strategy G (except at $t = 0$). As we have shown in Section 4.1 this is the only kind of strategy played in equilibrium.
■

Proof of Proposition 10. The proof closely follows the logic of the arguments used in the general game. Here we concentrate on the points where some adjustments are needed. We begin with the derivation of the complete information payoffs, i.e. the terminal payoffs in the signaling game. If $\pi_t = 1$, then the entrant solves the optimal stopping problem

$$W(x) = \sup_{t \leq \tau \leq \infty} E\left[\int_{\tau}^{\infty} e^{-r(u-t)} D_2^w X_u du - e^{-r(\tau-t)} K | X_t = x\right].$$

In the continuation region, i.e. for $x \in (0, U(1))$, $W(x)$ satisfies the following Bellman-type differential equation

$$rW(x) = \mu x W'(x) + \frac{1}{2}\sigma^2 x^2 W''(x) + D_2^w x$$

with the following three boundary conditions $W(U(1)) = d_2U(1) - K$, $W'(U(1)) = d_2$ and $W(0) = 0$. Using these in the general solution to the differential equation yields $W(x)$ and $U(1)$ preceding the proposition. $U(0)$ can be found in a straight-forward way. The value of the incumbent of w type $\Omega(x)$ satisfies a similar differential equation in $x \in (0, U(1))$, that is

$$r\Omega(x) = \mu x\Omega'(x) + \frac{1}{2}\sigma^2 x^2 \Omega''(x) + M^w x,$$

subject to $\Omega(U(1)) = D_1^w U(1)/(r-\mu)$ and $W(0) = 0$. The formula for $\Omega(x)$ given above the proposition follows.

Next, following Lemma 5, we characterize the condition on $U(m)$ such that $m \in \operatorname{supp}(G)$. Let $F(x, m)$ be the value function of the incumbent of the w type satisfying the condition that the firm is indifferent between stopping signaling and continuing at $x = m$ for all $m \in \operatorname{supp}(G)$. In the continuation region, for $x \in (m, \tilde{U}(m))$ with m fixed, $F(x, m)$ must satisfy

$$rF(x,m) = \mu x F_x(x,m) + \frac{1}{2}\sigma^2 x^2 F_{xx}(x,m) - M^p x,$$

subject to the continuous and smooth fit conditions $F(\tilde{U}(m), m) = D_1^w \tilde{U}(m)/(r-\mu)$, $F(m,m) = \Omega(m)$, $F_x(m,m) = \Omega'(m)$, and the normal reflection condition $F_m(m,m) = 0$ for all $m \in \operatorname{supp}(G)$. Solving the system of boundary conditions with the general solution we obtain

$$\begin{aligned}\{(\beta_1-\beta_2)(r-\mu)\Omega(m) - [(\beta_1-1)M^w + (1-\beta_2)M^p]\, m\} \left(\frac{\tilde{U}(m)}{m}\right)^{\beta_1} \\ + (\beta_1-1)(M^w - M^p)\, m \left(\frac{\tilde{U}(m)}{m}\right)^{\beta_2} + (\beta_1-\beta_2)(M^w - M^p)\,\tilde{U}(m) = 0.\end{aligned}$$

After differentiating this implicit equation in $\tilde{U}(m)$ with respect to m we obtain (30). If in the same problem we use a boundary condition for $m = L_1$ at the upper trigger as $F(U(0), L_1) = D_1^w U(0)/(r-\mu)$, we obtain equation (31) defining L_1.

At the next step we derive the best response of the entrant to a continuous strategy G of the incumbent. Similar to Section 4.2 denote the value of the entrant in this best response problem by $V(x,m)$. $V(x,m)$ must satisfy the following differential equation

$$rV(x,m) = \mu x V_x(x,m) + \frac{1}{2}\sigma^2 x^2 V_{xx}(x,m).$$

The boundary conditions of continuous and smooth fit are $V(\tilde{U}(m), m) = \hat{\Pi}(m) d_2 \tilde{U}(m) - K$ and $V_x(\tilde{U}(m), m) = \hat{\Pi}(m) d_2$. The boundary condition at the diagonal (m,m) is $V_m(m,m) = \Pi(m) g(m) (V(m,m) - W(m))$. After solving the set of boundary conditions with the solution to the differential equation and reorganizing we obtain (29) in the proposition, where, as before, $\Delta_1 = (m/\tilde{U}(m))^{\beta_1} - (m/\tilde{U}(m))^{\beta_2}$ and $\Delta_2 = \beta_2 (m/\tilde{U}(m))^{\beta_1} - \beta_1 (m/\tilde{U}(m))^{\beta_2}$.

The reminder of the proof is identical to the proof of Proposition 9. ■

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