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Characterizing QALYs under a General Rank Dependent Utility Model

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Abstract

This paper provides a characterization of QALYs, the most important outcome measure in medical decision making, in the context of a general rank dependent utility model. We show that both for chronic and for nonchronic health states the characterization of QALYs depends on intuitive conditions. This facilitates the assessment of the validity of QALYs in rank dependent non-expected utility theories and a comparison with other utility based measures of health.

Key words: utility theory, choice under uncertainty, value of life

JEL Classification: D8, I10

In many public policy decision contexts no good market data exist to value health outcomes. In these circumstances, most studies have relied on surveys to derive values for health outcomes (for an overview see Viscusi (1993)). The commonly used methodology in surveys has been to ask members of a representative sample how much they are willing to pay for reductions in the risk of contracting a particular disease. This methodology is referred to as "contingent valuation." Over the past decade, there has been an increasing awareness of potential problems with the contingent valuation approach (e.g. Hausman (1993)). As a result of these problems, several studies have suggested to use utility based measures of health outcomes (Viscusi et al. (1991), Krupnick and Cropper (1992), Jones-Lee et al. (1995), Magat et al. (1996)). The theoretical relationship between the contingent valuation method and utility based measures is straightforward (Viscusi et al., 1991), but empirical evidence suggests that the latter are more reliable.

The utility based approach to the valuation of health has been frequently applied in allocative decisions in health care (Torrance (1986)). The most important utility model in health decision making is a simple additive model: the quality-adjusted life-years (QALY) model. According to the QALY model the utility of a stream of health outcomes is calculated by multiplying each year of life by a weight reflecting the utility of the health state in which this year is spent. A major advantage of the QALY model is that it is easy to use. On the other hand, the QALY model can only be applied if individual preferences satisfy certain restrictions. Insight in these restrictions is important both for an assessment of the validity of the model and for a comparison of QALY based analyses with other utility based health measures and with studies using the contingent valuation method.

Existing characterizations of the QALY model all assume expected utility theory. It has now become widely acknowledged that expected utility theory is not valid as a descriptive theory of decision under risk. A number of the classic counterexamples to expected utility theory deal with cases in which outcomes are expressed in terms of health status and life or death. Among the alternative theories that have been proposed, the rank dependent utility theories of decision under risk and uncertainty (Quiggin (1981,1982), Yaari (1987), Schmeidler (1989)) and their derivative, cumulative prospect theory (Tversky and Kahneman (1992)) are currently the most important. Rank dependent utility models generalize expected utility theory by not only transforming outcomes to utilities, but also probabilities to decision weights π, which are monotonic, but need not be additive, i.e. $\pi(a + b) \neq \pi(a) + \pi(b)$ can happen. A correction for probability weighting is made to allow for the common empirical finding that individuals put too much weight on small and very large probabilities and not enough on probabilities in the middle range. We describe rank dependent utility theories in some more detail in section 3.

The aim of this paper is to provide a characterization of QALYs in the context of a general class of rank dependent utility models axiomatized by Miyamoto (1988). We impose a basic additive structure on a rank ordered set and we use this basic additive structure to provide a characterization of QALYs both for the situation where health status is constant over time, i.e. chronic health states, and for the situation where health status can vary over time.

In what follows, section 1 provides a brief overview of characterizations of QALYs that assume expected utility theory to hold. For chronic health states characterizations are available from the literature. We provide a characterization for nonchronic health states. In section 2 we present the general rank dependent utility model and characterize QALYs within this model. We show that if health states are chronic, then the central condition of QALYs is constant marginal utility for life years. The concept of marginal utility is widely used both in economics and in decision theory. For nonchronic health states, the central condition is generalized marginality. Generalized marginality is a natural extension of utility independence, a condition well-known from the decision theory literature. It is the strength of our characterizations that QALYs can be characterized in terms of conditions that are both relatively easily understood and straightforward to test empirically. This facilitates an assessment of the validity of QALYs in a general rank dependent class of utility models and a comparison with alternative valuations of health outcomes. In section 3 we show that rank dependent utility theory, Choquet expected utility theory and cumulative prospect theory are consistent with the basic additive structure we employ in this paper. Section 4 concludes the paper. Proofs are left for the appendix.

1. Characterizations of QALYs under expected utility theory

1.1. Notation and structural assumptions

We study a set of simple probability distributions (lotteries) over a set of health profiles (i.e. sequences of health outcomes). A typical element of the set of lotteries is

$(p^1,h^1;\ldots;p^m,h^m)$, where h^i stands for health profile i and p^i is the corresponding probability. A preference relation $\succsim$, meaning "at least as preferred as", is defined over the set of lotteries. Denote the symmetric part of $\succsim$ (indifference) by $\sim$ and the asymmetric part (strict preference) by $\succ$. Restricted to degenerate lotteries $\succsim$ is a preference relation over the set of health profiles H. We assume that $\succsim$ satisfies the von Neumann Morgenstern axioms (von Neumann and Morgenstern (1944)). Then $\succsim$ can be represented by the expected utility model: $EU(p^1,h^1;\ldots;p^m,h^m)=\Sigma_{i=1}^{m}p^iU(h^i)$, where U: $H \rightarrow \mathbb{R}$ is a real-valued utility function unique up to positive linear transformations.

1.2. Chronic health states

If health status is constant, then the set of outcomes consists of elements (Q,T), where Q stands for (constant) health status and T for the number of life years. Several structural assumptions are made with respect to the set of outcomes. These are given in the appendix.

The best known characterization of the QALY model for chronic health states is Pliskin, Shepard and Weinstein (1980). Pliskin et al. show that the QALY model follows if utility independence, risk neutrality on life years and constant proportional tradeoffs for two health states are imposed in addition to the axioms of expected utility theory. Utility independence and risk neutrality are well known conditions in decision theory (cf. e.g. Keeney and Raiffa (1976)). Constant proportional tradeoffs holds for two health states Q^1 and Q^2 if the proportion of life-years an individual is willing to give up for an improvement in health is invariant with respect to the number of life years: $U(Q^1,T) = U(Q^2,qT)$ for positive q and all T.

Bleichrodt, Wakker and Johannesson (1996) have shown that utility independence and constant proportional tradeoffs can be dispensed with in the presence of a condition that is entirely self-evident for health outcomes, the zero-condition, which asserts that for a time duration of zero life years all quality of life levels are equivalent. This leaves risk neutrality on life years as the central condition in the characterization of the QALY model.

Both in Pliskin et al. and in Bleichrodt et al. the set of health states is confined to positive health states, that is health states preferred to death. Miyamoto et al. (1996) show that the QALY model can still be characterized by risk neutrality on life years and the zero-condition if the set of health states is general and includes zero and negative health states in addition to positive health states.

1.3. Nonchronic health states

For nonchronic health states, the set of outcomes H consists of profiles $(q_1,\ldots q_T)$ where q_t stands for health status in period t and the utility function thus contains T attributes: $U = U(q_1,\ldots,q_T)$. In comparison with section 1.2 time is no longer continuous, but discrete. This does not make the two approaches incomparable. If we take a sufficiently fine discrete model, i.e. the length of the different time periods approaches zero, then the discrete time model will approach the continuous time model. Structural assumptions with respect to H are given in the appendix.

The first step in the characterization of QALYs is to impose additive independence (Fishburn (1965)). Additive independence holds if preferences over lotteries on $q_1,\ldots,q_T$ depend only on their marginal probability distributions and not on their joint probability distribution. Theorem 4 in Fishburn (1965) shows that given additive independence $U(q_1,\ldots,q_T)$ is equal to $\Sigma_t U_t(q_t)$. Continuity of the preference relation over H gives continuity of the one-period utility functions (Maas and Wakker (1994)). The characterization of QALYs is completed by adding a symmetry condition, that makes the preference relation invariant with respect to the point in time at which an outcome occurs: all time periods have equal weight.

Symmetry

$$(q_1,\ldots,q_T) \sim (q_{\pi(1)},\ldots,q_{\pi(T)}) \quad \text{for all health profiles } (q_1,\ldots,q_T) \text{ and permutation function } \pi(t).$$

where a permutation function π(t) is a function specifying a rearrangement of the time periods. The preceding exposition is summarized in theorem 1, a proof of which is given in the appendix.

Theorem 1

The following two statements are equivalent under expected utility theory and the structural assumptions made:

(i) the individual preference relation satisfies additive independence and symmetry.
(ii) The QALY model holds: $U(q_1,\ldots q_T) = \Sigma_t U(q_t)$, where U is continuous and unique up to positive linear transformations.

2. Characterizing QALYs under a general rank dependent utility model

2.1 Assumptions

We assume that there are two states of nature r and s. Probabilities for these states may but need not be known. H is the set of health profiles and $H^2 = H \times H$ is a set of acts. An act is a function from the set of states of nature to the real numbers, i.e. $f = (f_r, f_s) \in H^2$ is the act giving f_r if r is the true state and f_s if s is the true state. A preference relation $\succsim$ is defined on H^2, meaning "at least as preferred as." We assume that $\succsim$ is a weak order: $\succsim$ is transitive and complete. The relation $f \succ g$ (strict preference) is defined as $f \succsim g$ and not $g \succsim f$. The relation $f \sim g$ (indifference) is defined as both $f \succsim g$ and $g \succsim f$. Preferences under certainty are derived from preferences over constant acts: for $x,y \in H$, $x \succsim y$ iff (x,x)

$\succsim (y,y)$. The set of rank ordered acts $H^2_\downarrow$ is defined as the set $\{f \in H^2 : f_r \succsim f_s\}$ that is all acts for which the outcome under state r is at least as preferred as the outcome under state s. This set contains all constant acts.

Various non-expected utility theories, defined over the set of rank ordered acts, assume that the preference relation $\succsim$ on $H^2_\downarrow$ is additive, i.e. that there exist functions $U_r{:}H \rightarrow \mathbb{R}$ and $U_s{:}H \rightarrow \mathbb{R}$ such that $U(f) = U_r(f_r) + U_s(f_s)$ represents $\succsim$ on $H^2_\downarrow$. Expected utility theory is the special case where $U_r(f_r) = p_r U(f_r)$ and $U_s(f_s) = p_s U(f_s)$, with the decision weights p_r and p_s equal to the (subjective) probabilities. Non-expected utility theories do not necessarily use (subjective) probabilities as decision weights.

We impose preference conditions on the preference relation $\succsim$ on $H^2_\downarrow$ such that it can be represented by an additive function. These conditions are given in the appendix. It follows from these conditions that U_r and U_s are both continuous and preserve $\succsim$ on H. The assumptions we subsequently use in the derivation of QALYs ensure that U_r and U_s are linear with respect to each other and thus can be rewritten as $U_r = \pi_r U$ and $U_s = \pi_s U$, with π_r and π_s scaling constants or decision weights that sum to one and that may but need not be interpreted as probabilities.

Following Wakker (1984), we define a preference relation $\succsim^*$ in the following way:

$$[a;b] \succsim {}^*[c;d] \text{ if}$$

$$(a,z) \succsim (b,y) \text{ and } (c,z) \precsim (d,y) \text{ and/or } (s,a) \succsim (t,b) \text{ and } (s,c) \precsim (t,d)$$

$$\text{for all } a,b,c,d,s,t,y,z \in H \text{ and } (a,z),(b,y),(c,z),(d,y),(s,a),(t,b),(s,c),(t,d) \in H^2_\downarrow.$$

The preference relation $\succsim^*$ can be interpreted to measure strength of preference. To see this suppose $(a,z) \succsim (b,y)$ and $(c,z) \precsim (d,y)$. Given the additive structure, $U_r(a) + U_s(z) \geq U_r(b) + U_s(y)$ and $U_r(c) + U_s(z) \leq U_r(d) + U_s(y)$. Or $U_r(a) - U_r(b) \geq U_s(y) - U_s(z) \geq U_r(c) - U_r(d)$. Evaluated by utility function U_r the strength of preference of a over b is at least as great as the strength of preference of c over d. At this stage we cannot exclude the possibility that evaluated by utility function U_s the strength of preference of c over d is greater than the strength of preference of a over b, because the additive representation does not ensure that U_r and U_s are linear with respect to each other.

We define $[a;b] \succ^* [c;d]$ if one of the above inequalities holds strict (i.e. one of the $\succsim$ signs can be replaced by a $\succ$ sign or one of the $\precsim$ signs can be replaced by a [$\prec$]sign).

2.2. Chronic health states

Because the QALY model is linear in life years, we have to establish linearity of $U(T)$. If expected utility theory holds, linearity of the utility function follows from risk neutrality. This characterization is possible, because probabilities are evaluated linearly, that is the decision weights π_r and π_s are equal to the (subjective) probabilities of state s and state t. In the general rank dependent utility model we have not restricted the decision weights to be equal to the respective probabilities and therefore a different condition has to be imposed.

One reason expected utility theory has been criticized is that both risk attitude and attitudes towards outcomes are modelled through the utility function and therefore cannot be distinguished. Most rank dependent generalizations separate attitudes towards probabilistic risk, which are modelled through the decision weights, from attitudes towards outcomes, which are modelled through the utility function. Because we want to establish linearity of the utility function, we have to use a condition about attitude towards outcomes. In our derivation we use constant marginal utility:

Constant marginal utility

The preference relation $\gtrsim$ on $H^2_\downarrow$ satisfies *constant marginal utility* if $[a + e;b + e] >^* [a;b]$ is excluded for all $a,b,a + e,b + e \in H$ and all $e \in \mathbb{R}$.

By constant marginal utility, if two outcomes are increased or decreased by the same amount then this should have no effect on the strength of preference of one outcome over the other. The implications of constant marginal utility are strong. In the appendix we prove the following theorem:

Theorem 2

If the general rank dependent utility model $U(f) = U_r(f_r) + U_s(f_s)$ and the structural assumptions hold, then the following two statements are equivalent:

(i) Constant marginal utility holds on $H^2_\downarrow$
(ii) There exists a continuous utility U, linear in life years, and unique up to positive linear transformations and decision weights π_r and π_s such that $\gtrsim$ on $H^2_\downarrow$ is represented by $U(x,y) = \pi_r U(x) + \pi_s U(y)$.

We can now use a similar line of argument as in Bleichrodt et al. (1996) and in Miyamoto et al. (1996) to derive the QALY model in the general rank dependent utility framework. By constant marginal utility for life years, holding quality of life fixed, $U(Q,T)$ takes the form $C(\text{Q}) + V(Q)*T$, where $C(Q)$ and $V(Q)$ are constants that depend on Q but not on T. Clearly this form is not yet equivalent to the QALY model, because in the QALY model all quality of life levels are equivalent for a life duration of zero life years: $U(Q,0)$ is constant for all Q. This can be ensured by imposing the zero-condition. By the zero-condition, $U(Q,0) = C(Q) + V(Q)*0 = C(Q)$ is constant. Finally we use the uniqueness property of the utility function U, and subtract C(Q) to obtain $U(Q,T) = V(Q)*T$, which is the QALY model.

To summarize, we have proved the following theorem:

Theorem 3

If the general rank dependent utility model $U(f) = U_r(f_r) + U_s(f_s)$ and the structural assumptions hold, then the following two statements are equivalent:

(i) the preference relation satisfies constant marginal utility on life years for each health state and the zero-condition.
(ii) the QALY model $U(Q,T) = V(Q)$*T holds with U and V continuous and unique up to positive linear transformations.

2.3. Nonchronic health states

By $y_t x$ we denote a health profile x with health state x_t replaced by y_t: $(x_1,.., x_{t-1}, y_t, x_{t+1},\ldots, x_T)$. Let $x_t\, y_{t+i}\, z$ denote a health profile z with z_t replaced by x_t and z_{t+i} by y_{t+i}, and let $y_A x$ be a health profile with elements equal to y for all $t \in A$ and equal to x for all $t \notin A$.

Utility independence in the general utility model is defined as follows:

Utility Independence

H^J $(J \subset N)$ is *utility independent* on $H^2_\downarrow$ if $(k_J x, l_J x) \gtrsim (m_J x, n_J x) \Leftrightarrow (k_J y, l_J y) \gtrsim (m_J y, n_J y)$ when all acts are contained in $H^2_\downarrow$. *Utility independence* holds if H^J is utility independent on $H^2_\downarrow$ for every $J \subset N$.

Utility independence gives linearity of U_r and U_s with respect to each other and ensures that U is either additive or multiplicative (Miyamoto and Wakker (1996)). If expected utility theory holds, additive independence distinguishes between the additive and the multiplicative specification of U. Additive independence is not available as a diagnostic tool in the general rank dependent utility model, because it assumes linearity in probability. To distinguish between the additive and the multiplicative model, we use the following strengthening of utility independence, which generalizes a condition proposed by Miyamoto (1988) for two-attribute utility functions.

Generalized marginality

The preference relation $\gtrsim$ on $H^2_\downarrow$ satisfies *generalized marginality* if for all acts in $H^2_\downarrow$, for all $z, x_t, y_t, v_t, w_t, a_{t+i}, b_{t+i}, c_{t+i}, d_{t+i}$, and for all $t, t+i$ the following holds:

$$(x_t\, a_{t+i}\, z, y_t b_{t+i}\, z) \gtrsim (v_t\, a_{t+i}\, z, w_t\, b_{t+i}\, z) \Leftrightarrow (x_t\, c_{t+i}\, z, y_t\, d_{t+i} z) \gtrsim (v_t\, c_{t+i}\, z, w_t\, d_{t+i}\, z)$$

If $a_{t+i} = b_{t+i}$ and $c_{t+i} = d_{t+i}$ then it is easily seen that generalized marginality is equal to utility independence for single attributes. Thus generalized marginality can be interpreted as a strengthening of utility independence for single attributes.

Theorem 4

If the general utility model $U(x,y) = U_r(x) + U_s(y)$ and the structural assumptions hold, then the following two statements are equivalent:

(i) The preference relation satisfies generalized marginality and symmetry
(ii) The QALY model holds: $U(q_1,\ldots,q_T) = \Sigma_t U(q_t)$ with U continuous and unique up to positive linear transformations.

A proof of theorem 4 can be found in the appendix.

The proof of theorem 4 shows that imposing generalized marginality in the general rank dependent utility model is necessary and sufficient to lead to an additive model: $U(q_1,\ldots,q_T) = \Sigma_t U_t(q_t)$. Additive models satisfy utility independence for all subsets. Thus, generalized marginality, which we motivated as a strengthening of utility independence for single attributes, implies utility independence in full.

Corollary 1

If the general utility model is of the form $U(x,y) = U_r(x) + U_s(y)$, and the structural assumptions given in the appendix hold, then generalized marginality implies utility independence.

If utility independence holds, but not generalized marginality, then U is multiplicative: $U(q_L,\ldots,q_T) = \Pi_t U_t(q_t)$ (Miyamoto and Wakker (1996)). It is easy to show using the proof of theorem 1 that the U_t are identical if symmetry is imposed on top of utility independence.

3. Compatibility of the general utility model with rank dependent utility theory and cumulative prospect theory

We show in this section that the general rank dependent utility model is compatible with rank dependent utility theory (Quiggin (1981)), Choquet expected utility theory (Schmeidler (1989)), and cumulative prospect theory (Tversky and Kahneman (1992)), currently the most influential alternatives for expected utility theory. It is important to derive that the general rank dependent utility model encompasses these theories as special cases, because it follows by implication that the results derived in section 2 are valid under these theories.

3.1. Rank dependent utility theory

In rank dependent utility theory preferences are defined over rank-ordered lotteries $(p^1,h^1;\ldots;p^m,h^m)$: $h^l \gtrsim \ldots\ldots \gtrsim h^m$. Rank ordered lotteries are evaluated by:

$$\sum_{j=1}^{m}[w(\sum_{k=1}^{j}p_k) - w(\sum_{k=1}^{j-1}p_k)]U(h^k) \tag{1}$$

where $U{:}H\rightarrow\mathbb{R}$ is a continuous utility and $w{:}[0,1]\rightarrow[0,1]$ is a continuous, strictly increasing function such that $w(0) = 0$ and $w(1) = 1$.

For any p, the rank dependent utility representation of a lottery $(p,h^1;(1-p),h^2) \in H^2_\downarrow$, $w(p)U(h^1) + [1 - w(p)]U(h^2)$, is an additive representation on $H^2_\downarrow$ and thus our results apply within this structure, yielding characterizations of QALYs both for chronic and for nonchronic health states.

3.2. Choquet expected utility theory

Choquet expected utility theory is the application of rank dependent utility theory to decision making under uncertainty, i.e. probabilities are no longer given. In Choquet expected utility theory, a continuous utility $U{:}H{\rightarrow}\mathbb{R}$ is selected along with a capacity W. Let S be the set of states of nature. We assume for simplicity that S is finite. 2^s is the set of all subsets of S. A capacity is a function $W{:}2^s{\rightarrow}\mathbb{R}$ such that $W(\emptyset) = 0$, $W(S) = 1$, and $A \subseteq B \Rightarrow W(A) \leq W(B)$. $f(s_1)$ stands for the outcome of act f if s_1 is the true state of nature. To apply Choquet expected utility acts have to be rank ordered: $f(s_1) \gtrsim ...... \gtrsim f(s_m)$. Rank ordered acts are evaluated by:

$$\sum_{j=1}^{m}[W(s_1,\ldots,s_j) - W(s_1,\ldots,s_{j-1})]U(f(s_j)) \tag{2}$$

Let $A \subset S$ and denote the complement of A by A^C. The Choquet expected utility representation of an act $(A,h^1;A^C,h^2) \in H^2_\downarrow$, $W(A)U(h^1) + [1 - W(A)]U(h^2)$, is an additive representation on $H^2_\downarrow$ and thus our results apply within this structure.

3.3. Cumulative prospect theory

Decision under risk

In Tversky and Kahneman's (1992) cumulative prospect theory the evaluation of a gamble depends not only on the ranks of the outcomes but also on their signs: outcomes are evaluated differently depending on whether they are gains or losses. The sign of an outcome depends on its position relative to a reference outcome. For our analysis we can restrict attention to the set of pure gain lotteries $G^2_\downarrow$ containing all lotteries $(p^1,h^1;\ldots;p^m,h^m)$ such that $h^1 \gtrsim ...... \gtrsim h^m \gtrsim r$, where r denotes the reference health profile and to the set of pure loss lotteries $L^2_\downarrow$ containing all lotteries $(p^1,h^1;\ldots;p^m,h^m)$ such that $r \gtrsim h^1 \gtrsim ...... \gtrsim h^m$. Pure gain lotteries are evaluated as in rank dependent utility theory.

$$\sum_{j=1}^{m}[w^+(\sum_{k=1}^{j}p_k) - w^+(\sum_{k=1}^{j-1}p_k)]U(h^k),(p^1,h^1;\ldots;p^m,h^m)\in G^2_\downarrow \tag{3}$$

Pure loss lotteries are evaluated by:

$$\sum_{j=1}^{m}[w^-(\sum_{k=j}^{m}p_k) - w^\pm(\sum_{k=j+1}^{m}p_k)]U(h^k),(p^1,h^1;\ldots;p^m,h^m)\in L^2_\downarrow \tag{4}$$

where $U{:}H \rightarrow \mathbb{R}$ is a continuous utility and both $w^{+}{:}[0,1] \rightarrow [0,1]$ and $w^{-}{:}[0,1] \rightarrow [0,1]$ are continuous, strictly increasing functions such that $w^{+}(0) = w^{-}(0) = 0$ and $w^{+}(1) = w^{-}(1) = 1$.

For any p, the cumulative prospect theory representation of a pure gain lottery $(p,h^1;(1-p),h^2) \in G^2_{\downarrow}$, $w^{+}(p)U(h^1) + [1 - w^{+}(p)]U(h^2)$, is an additive representation on $G^2_{\downarrow}$ and the representation of a pure loss lottery lottery $(p,h^1;(1-p),h^2) \in L^2_{\downarrow}$, $w^{-}(p)U(h^2) + [1 - w^{-}(p)]U(h^1)$, is an additive representation on $L^2_{\downarrow}$. Therefore, our results apply within these structures, yielding characterizations of QALYs both for chronic and for nonchronic health states.

Decision under uncertainty

In the situation of decision making under uncertainty, cumulative prospect theory generalizes Choquet expected utility theory by introducing sign dependence. There are now two, not necessarily equal, capacities $W^{+}{:}2^S \rightarrow \mathbb{R}$ and $W^{-}{:}2^S \rightarrow \mathbb{R}$. Rank ordered pure gain acts $f(s_1) \gtrsim \ldots\ldots \gtrsim f(s_m) \gtrsim r$ are evaluated by

$$\sum_{j=1}^{m}[W^{+}(s_1,\ldots,s_j) - W^{+}(s_1,\ldots,s_{j-1})]U(f(s_j)) \tag{5}$$

and rank ordered pure loss acts $r \gtrsim f(s_1) \gtrsim \ldots\ldots \gtrsim f(s_m)$ by

$$\sum_{j=1}^{m}[W^{-}(s_j,\ldots,s_m) - W^{-}(s_{j+1},\ldots,s_m)]U(f(s_j)) \tag{6}$$

where $U{:}H \rightarrow \mathbb{R}$ is a continuous utility.

Let $A,B \subset S$ and denote their complements by A^C and B^C. The cumulative prospect theory representation of an act $(A,h^1;A^C,h^2) \in G^2_{\downarrow}$, $W^{+}(A)U(h^1) + [1 - W^{+}(A)]U(h^2)$, is an additive representation on $G^2_{\downarrow}$ and the cumulative prospect theory representation of an act $(B,h^1;B^C,h^2) \in L^2_{\downarrow}$, $W^{-}(B)U(h^2) + [1 - w^{-}(B)]U(h^1)$, is an additive representation on $L^2_{\downarrow}$. Our results apply within these structures.

4. Concluding remarks

In this paper we have characterized the QALY model in the context of a general rank dependent utility model, which is consistent with the most important non-expected utility models. We have shown that the crucial conditions are constant marginal utility for life years in the case of chronic health states, and generalized marginality, an extension of utility independence for single attributes, in the case of nonchronic health states. It is the strength of our characterization that even though expected utility theory is replaced by a more general rank dependent utility model, the measurement foundation of QALYs is easily understood and straightforward to test empirically.

In the derivation of our main results we assumed that there are only two states of nature. It is straightforward to extend the analysis to more than two states of nature. Similarly, our results can be generalized to other contexts than medical decision making. If one of the attributes in a two-attribute utility function corresponds directly to the set of real numbers, i.e. the utility function for this attribute is a function from $\mathbb{R}$ to $\mathbb{R}$, and a zero-condition can be used, then constant marginal utility for attribute y characterizes the utility function $U(x,y) = U(x)*y$. Generalized marginality can be used in any context and does not require any of the attributes to correspond directly to the set of real numbers or a zero-condition to hold.

Appendix: Structural assumptions and proofs

A1: Structural assumptions

Chronic health states

The set of life years T is a subset of the set of real numbers and therefore endowed with the Euclidean topology which is connected and separable. The set of health states (quality of life levels) Q is a connected and separable topological space. The set of chronic health profiles $H = Q \times T$ is a Cartesian product of the set of life years and the set of health states, and is endowed with the product topology. The preference relation $\succsim$ on H is continuous: both $\{x \in H: x \succsim y\}$ and $\{x \in H: y \succsim x\}$ are closed for all $y \in H$. We further assume that both health state and life years are essential: there exist $Q^1,Q^2 \in Q$ and $T^1,T^2 \in T$ such that $(Q^1,T) \succ (Q^2,T)$ and $(Q,T^1) \succ (Q,T^2)$. Essentiality implies that both quality of life and quantity of life affect preferences over chronic health profiles.

Nonchronic health states

The set of health profiles H is a Cartesian product of $T \geq 2$ one period sets of health states: $Q_1 x Q_2 x .....x Q_T$. The sets of one period health states are connected topological spaces. They can be taken identical or different depending on whether all health states are attainable at all ages. The set H is endowed with the product topology. Denote by $x_t q$ health profile q with health state q_t replaced by health state x_t. The preference relation on H is continuous and all periods are essential: $x_t q \succ y_t q$ for at least two health states x_t and $y_t \in Q_t$ and for all t.

Assumptions for an additive representation on a rank ordered set:

The set of health profiles H is a connected topological space. The preference relation $\succsim$ on H derived from $\succsim$ on H^2 is continuous as is $\succsim$ on the set of rank ordered lotteries $H^2_{\downarrow}$. Further $\succsim$ on $H^2_{\downarrow}$ satisfies outcome monotonicity. That is, if $f_r \succsim g_r$ & $f_s \succsim g_s$ then

$f \gtrsim g$ with $f > g$ if at least one of the inequalities is strict. Finally we assume that $\gtrsim$ on $H^2_\downarrow$ satisfies the hexagon condition (Wakker (1989)): if $(y_1,a_2) \sim (x_1,b_2)$ & $(v_1,a_2) \sim (y_1,b_2)$ & $(y_1,b_2) \sim (x_1,c_2)$ then $(v_1,b_2) \sim (y_1,c_2)$.

By theorems 3.2 and 3.3, proposition 3.5 and remark 3.7 in Wakker (1993) an additive representation $U(f) = U_r(f_r) + U_s(f_s)$ represents $\gtrsim$ on the topological interior of $H^2_\downarrow$, where U_r and U_s are continuous functions, unique up to positive linear transformations, from the topological interior of H to $\mathbb{R}$. The representation can be extended to the whole of $H^2_\downarrow$ if U_r and U_s are linear with respect to each other.

A2: Proofs

Theorem 1

We prove that symmetry ensures that all one period utility functions are identical. By additive independence $U(q_1,\ldots,q_T) = U_1(q_1) + \ldots U_T(q_T)$. By symmetry $(q_1,\ldots,q_T) \sim (q_2,q_3,\ldots,q_T,q_1)$ which, given additive independence, is equal to $U_1(q_2) + \ldots + U_T(q_1)$. Further $(q_2,q_3,\ldots,q_T,q_1) \sim (q_3,q_4,\ldots,q_1,q_2) \sim \ldots (q_T,q_1,\ldots,q_{T-2},q_{T-1})$. Thus $U(q_1,\ldots,q_T) = U_1(q_1) + \ldots U_T(q_T) = U_1(q_2) + \ldots + U_T(q_1) = \ldots = U_1(q_T) + \ldots + U_T(q_{T-1})$. Or $U(q_1,\ldots,q_T) = 1/T\, \Sigma_t U_t(q_1) + \ldots + 1/T\, \Sigma_t U_t(q_T)$. Thus all one period utility functions are identical.

Theorem 2

That (ii) implies (i) is straightforward. So suppose (i) holds. First we derive that U_r and U_s are linear with respect to each other. Both U_r and U_s are continuous and preserve the ordering over H. Thus U_r and U_s are related by a continuous nondecreasing transformation ϕ: $U_s = \phi(U_r)$. Linearity of U_r and U_s with respect to each other follows by showing that ϕ is affine. Fix health status at a particular level. In the remainder of the proof we suppress this health status level. Thus the arguments in the functions refer to number of life years.

Take an arbitrary element $U_r(\zeta)$ from the domain of ϕ. There exists an open interval M around $U_r(\zeta)$ so small that, for all $U_r(a)$, $U_r(b)$ in M, there are x and y such that

$$U_r(a) - U_r(b) = U_s(y) - U_s(x) \tag{A1}$$

By continuity of ϕ we can further take M such that for all $U_r(a)$, $U_r(b)$ there are v and w such that

$$U_s(a) - U_s(b) = \phi(U_r(a) - U_r(b)) = U_r(w) - U_r(v) \tag{A2}$$

Now choose e such that $U_r(a + e)$ is in M and $b + e = a$. By constant marginal utility

$$U_r(a + e) - U_r(b + e) = U_r(a + e) - U_r(a) = U_r(a) - U_r(b). \tag{A3}$$

But by constant marginal utility $U_s(a + e) - U_s(b + e)$ must also be equal to $U_s(a) - U_s(b)$, because if for example $U_s(a + e) - U_s(b + e) > U_s(a) - U_s(b)$ then this would imply that $[a + e; b + e] \succ^* [a;b]$ which is excluded by the assumption of constant marginal utility.

Thus $U_s(a + e) - U_s(a) = U_s(a) - U_s(b)$ or $\phi(U_r(a + e) - U_r(a)) = \phi(U_r(a) - U_r(b))$ and so on M ϕ satisfies Jensen's equality: $\phi((a + b)/2) = \phi(a)/2 + \phi(b)/2$. By corollary A1.3 in Wakker (1989) this establishes that ϕ is affine and thus $U_s = a_s + bU_r$, $a_s \in \mathbb{R}$, $b > 0$. Set U equal to U_r, and $\pi_r = 1/(1 + b)$ and $\pi_s = b/(1 + b)$. This establishes the representation $U(x,y) = \pi_r U(x) + \pi_s U(y)$. U is continuous and unique up to positive linear transformations because U_r is.

Linearity of U in T follows from corollary 16 in Wakker (1994) by noting that all that is needed there is the representation $U(x,y) = \pi_r U(x) + \pi_s U(y)$ with U unique up to positive linear transformations and the decision weights uniquely determined.

Theorem 4

That (i) implies (ii) is routine. Suppose (ii). To derive is (i). First we establish linearity of U_r and U_s with respect to each other. For $T = 2$ it is easy to see that generalized marginality implies utility independence by setting either $x_t = y_t$ and $v_t = w_t$ or $a_{t+i} = b_{t+i}$ and $c_{t+i} = d_{t+i}$. Utility independence in combination with the structural assumptions gives linearity of U_r and U_s with respect to each other (Miyamoto and Wakker (1996)). Let $T > 2$. In the proof of their theorem 4 Miyamoto and Wakker only need an additive representation on H and utility independence for one coordinate. Both are ensured by generalized marginality. Generalized marginality implies weak separability and independence of equal subalternatives with subalternatives of length $n - 2$ and thus by the proof of theorem III.4.1. in Wakker (1989) the additive representation follows. Utility independence for one coordinate follows by setting $a_{t+i} = b_{t+i}$ and $c_{t+i} = d_{t+i}$. By a similar line of proof as in Miyamoto and Wakker (1996) it can be shown that U_r and U_s are linear with respect to each other and $U(x,y)$ can be represented by $\pi_r U(x) + \pi_s U(y)$. U is continuous and unique up to positive linear transformations because both U_r and U_s are. Moreover U is either multiplicative or additive. The multiplicative case is ruled out by generalized marginality. Suppose U is multiplicative. The pattern of preferences $(x_t\, a_{t+i}\, z, y_t\, b_{t+i}\, z) \succsim (v_t\, a_{t+i}\, z, w_t\, b_{t+i}\, z)$ implies, after division by common terms, that $\pi_r U_t(x_t)U_{t+i}(a_{t+i}) + \pi_s U_t(y_t)U_{t+i}(b_{t+i}) \geq \pi_r U_t(v_t)U_{t+i}(a_{t+i}) + \pi_s U_t(w_t)U_{t+i}(b_{t+i})$ or that $\pi_r U_{t+i}(a_{t+i})[U_t(x_t) - U_t(v_t)] \geq \pi_s U_{t+i}(b_{t+i})[U_t(w_t) - U_t(y_t)]$. This does not imply that $\pi_r U_{t+i}(c_{t+i})[U_t(x_t) - U_t(v_t)] \geq \pi_s U_{t+i}(d_{t+i})[U_t(w_t) - U_t(y_t)]$ or that $(x_t\, c_{t+i}\, z, y_t\, d_{t+i}\, z) \succsim (v_t\, c_{t+i}\, z, w_t\, d_{t+i}\, z)$ which is required by generalized marginality. If we choose $U_{t+i}(c_{t+i})$ sufficiently small compared to $U_{t+i}(d_{t+i})$ preference will reverse and generalized marginality is violated. For additive U, $(x_t\, a_{t+i}\, z, y_t\, b_{t+i}\, z) \succsim (v_t\, a_{t+i}\, z, w_t\, b_{t+i}\, z)$ implies, after subtraction of common terms, that $\pi_r\, [U_t(x_t) + U_{t+i}(a_{t+i})] + \pi_s\, [U_t(y_t) + U_{t+i}(b_{t+i})] \geq \pi_r\, [U_t(v_t) + U_{t+i}(a_{t+i})] + \pi_s\, [U_t(w_t) + U_{t+i}(b_{t+i})]$ or $\pi_r[U_t(x_t) - U_t(v_t)] \geq \pi_s[U_t(w_t) - U_t(y_t)]$. The a_{t+i} and b_{t+i} terms vanish and can be replaced by any c_{t+i} and d_{t+i} terms without affecting preference and thus generalized marginality holds.

By generalized marginality $U(q_1,\ldots,q_T) = \Sigma_t U_t(q_t)$. Symmetry then ensures that all U_t are identical.

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