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Technical Appendix to

THE VALIDITY OF QALYS UNDER NON-EXPECTED UTILITY

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Appendix A: Characterisation of Expression (1) When the Outcome Set is Connected

Wakker (1993) characterised the additive representation $f \mapsto U_r(f_r) + U_s(f_s)$ for the case where the outcome set $\mathcal{H}$ is a connected topological space. If $\mathcal{H}$ is a topological space and $\mathcal{H}^2_\downarrow$ is endowed with the restriction of the product topology on $\mathcal{H}^2$ then $\succeq$ is *continuous* on $\mathcal{H}^2_\downarrow$ if for all $y \in \mathcal{H}$ the sets $\{x \in \mathcal{H}^2_\downarrow : x \succeq y\}$ and $\{x \in \mathcal{H}^2_\downarrow : x \preceq y\}$ are closed. The preference relation satisfies *outcome monotonicity* if for all acts f, g in $\mathcal{H}^2_\downarrow$ if $f_r \succeq g_r$ and $f_s \succeq g_s$ then $f \succeq g$, where the consequent preference is strict if either antecedent preference is strict. The *Thomsen condition* is satisfied on $\mathcal{H}^2_\downarrow$ if $(f_r, f_s) \sim (g_r, g_s)$ & $(h_r, g_s) \sim (f_r, h_t) \Rightarrow (h_r, f_s) \sim (g_r, h_t)$ whenever all six acts are contained in $\mathcal{H}^2_\downarrow$.

A health state $x \in \mathcal{H}$ is *maximal* if for no other health state $y \in \mathcal{H}$, $y \succ x$. A health state $x \in \mathcal{H}$ is *minimal* if for no other health state $y \in \mathcal{H}$, $x \succ y$. An *extreme act* either assigns to both states of nature r and s a maximal health state or to both states of nature r and s a minimal health state.

LEMMA 1 *Let $\mathcal{H}$ be a connected topological space, and let $\succeq$ be a weak order on $\mathcal{H}^2_\downarrow$ that satisfies continuity, outcome monotonicity and the Thomsen condition. Then there exist continuous functions U_r and U_s from $\mathcal{H}$ to $\mathbb{R}$ such that $f \mapsto U_r(f_r) + U_s(f_s)$ represents $\succeq$ on* $\mathcal{H}^2_\downarrow \backslash \{\text{extreme acts}\}$. *If U_r and U_s are linear with respect to each other, then the representation can be extended by continuity to the entire set $\mathcal{H}^2_\downarrow$. The functions U_r and U_s are unique up to a positive linear transformation with common units.*

Proof. See Wakker (1993, Theorem 3.3(a), Proposition 3.5, and Remark 3.7).

Appendix B: Extension to Outcome Sets that Are Not Connected

The set $\mathcal{Q}$ is general. Therefore, no topologies are naturally given on $\mathcal{H}$ and $\mathcal{H}^2$ and Lemma 1 no longer applies. However, Lemma 1 can be extended to a domain that is not connected, provided that the maximal connected subspaces of the domain, the topological components, overlap sufficiently in the preference order. The zero-condition, which was defined in Section 2, ensures this sufficient overlap.

If $\mathcal{H}$ is not a topological space, Wakker's definition of continuity is ambiguous. Instead, we assume that $\succeq$ is continuous in duration. For $j = r,s$, let $x_j f$ denote the prospect f with f_j replaced by x. The preference relation is *continuous in duration* if for all $f, g \in \mathcal{H}^2_\downarrow$, for $q \in \mathcal{Q}$ and for $j = r,s$, the sets $\{t \in \mathcal{T}: (q,t)_j f \succeq g\}$ and $\{t \in \mathcal{T}: (q,t)_j f \prec g\}$ are closed.

The set of durations $\mathcal{T}$ is an interval, and hence the Euclidean topology is defined on $\mathcal{T}$. It is well known that the Euclidean topology is connected.

LEMMA 2: *Let $\succeq$ be a weak order on $\mathcal{H}^2_\downarrow$ that satisfies the zero condition, monotonicity in duration, continuity in duration, outcome monotonicity, and the Thomsen condition on $\mathcal{H}^2_\downarrow$. Then there exist functions U_r and U_s from $\mathcal{H}$ to $\mathbb{R}$ such that $f \mapsto U_r(f_r) + U_s(f_s)$ represents $\succeq$ on $\mathcal{H}^2_\downarrow \backslash\{$extreme acts$\}$. U_r and U_s are strictly increasing in duration and continuous in duration. If U_r and U_s are linear with respect to each other, then the representation can be extended by continuity to the entire set $\mathcal{H}^2_\downarrow$. The functions U_r and U_s are unique up to a positive linear transformation with common units.*

Proof. Consider the order topology $\mathcal{T}_\succeq$ on $\mathcal{H}$, i.e., the smallest topology containing all sets $\{h \in \mathcal{H}: h \succ g\}$ and $\{h \in \mathcal{H}: h \succ g\}$. The preference relation $\succeq$ on $\mathcal{H}$ is continuous with respect to this topology. By Lemma 3.1 in Bleichrodt and Miyamoto (2003), $\mathcal{T}_\succeq$ is connected if the zero-condition holds. Then the product topology $\mathcal{T}^2_\succeq$ on the set of acts $\mathcal{H}^2$ is also connected. By Lemma 3.2 in Bleichrodt and Miyamoto (2003), $\succeq$ on $\mathcal{H}^2_\downarrow$ is continuous with respect to $\mathcal{T}^2_\succeq$ The preference relation $\succeq$ satisfies outcome monotonicity and the Thomsen condition. Hence, by Theorem 3.3 in Wakker (1993) there exist functions U_r and U_s from $\mathcal{H}$ to $\mathbb{R}$ such that $f \mapsto U_r(f_r) + U_s(f_s)$ represents $\succeq$ on $\mathcal{H}^2_\downarrow \backslash\{$extreme acts$\}$. U_r and U_s are continuous in duration by continuity in duration and strictly increasing in duration by monotonicity in duration. By Proposition 3.5 in Wakker (1993), if U_r and U_s are linear with respect to each other, then the representation can be extended to the entire set $\mathcal{H}^2_\downarrow$ by continuity of $\succeq$ with respect to $\mathcal{T}^2_\succeq$. By Theorem 3.3 in Wakker (1993), the functions U_r and U_s are unique up to a positive linear transformation with common units.

Appendix C: Generalisation to More Than Two States of Nature

Throughout the paper, we have assumed that there are only two states of nature. We now generalise our results to the case where the number of states of nature is arbitrary, but finite. Let $\mathcal{S} = \{1, \ldots, m\}$ be the state space. We assume that *outcome monotonicity* holds in the sense that if $f_j \succeq g_j$ for all $j \in \mathcal{S}$ then $f \succeq g$ with $f \succ$ g if at least one antecedent preference is strict. Hence, there are no null states. We further assume that there exist additive functions $V_j: \mathcal{H} \to \mathbb{R}$, $j \in \mathcal{S}$, such that $f \mapsto \Sigma_{j \in \mathcal{S}} V_j(f_j)$ represents $\succeq$ on $\mathcal{H}^{\mathrm{m}}_\downarrow$. For any proper subset $A \subset \mathcal{S}$, let (x, A, y) denote the act that gives outcome x if $s_j \in A$ and y otherwise. Let $\mathcal{H}^A_\downarrow$ be the set of all such acts with $x \succeq y$. $\mathcal{H}^A_\downarrow$ is isomorphic to $\mathcal{H}^2_\downarrow$ under the map $\varphi[(x, A, y)] = (x, y)$. Hence, the conditions identified in Theorems 1 and 2 can be applied to $\mathcal{H}^A_\downarrow$ to give the time-linear and the time-nonlinear QALY model, respectively. The implication that the QALY models imply the preference conditions on each $\mathcal{H}^A_\downarrow$, and in fact on the entire domain, are easy to verify and are left to the reader.

Appendix D: Proofs

Proof of Theorem 1 It is easily verified that (*ii*) implies (*i*). Suppose that (*i*) holds. Lemma 2 ensures that (1) holds. Hence, we can apply the proof of Theorem 2 in Bleichrodt and Quiggin (1997) to derive (*ii*).

Proof of Theorem 2. Suppose that (*ii*) holds. Because $V(q)$ is everywhere positive, it is easily verified that duration is utility independent of health status. The verification of the other conditions is straightforward.

Suppose that (i) holds. By Lemma 2 and continuity of $\succeq$ with respect to $T^2_{\succeq}$, the theorem holds in general if it holds in case $\mathcal{H}$ contains no maximal or minimal outcomes. Therefore, assume that $\mathcal{H}$ contains no maximal or minimal outcomes. Define $\mathcal{T}^2_\downarrow = \{(t_1, t_2) \in \mathcal{T}^2 : t_1 \geq t_2\}$. By monotonicity in duration, $(t_1, t_2) \in \mathcal{T}^2_\downarrow \Leftrightarrow ((q, t_1),(q, t_2)) \in \mathcal{H}^2_\downarrow$. Define the relation $\succeq_t$ on $\mathcal{T}^2_\downarrow$ by $(t_1, t_2) \succeq_t (t_3, t_4)$ if for some $q \in \mathcal{Q}((q, t_1),(q, t_2)) \succeq ((q, t_3),(q, t_4))$. Because duration is utility independent the choice of q is immaterial. Choose an arbitrary $q' \in \mathcal{Q}$ and define functions W_r and W_s from $\mathcal{T}$ to $\mathbb{R}$ by $W_r(t_1) = U_r(q', t_1)$ and $W_s(t_1) = U_s(q', t_1)$. Duration being utility independent on $\mathcal{H}^2_\downarrow$ implies that for any $q \in \mathcal{Q}$ both the function $(t_1,t_2) \mapsto U_r(q,t_1) + U_s(q,t_2)$ and the function $(t_1,t_2) \mapsto W_r(t_1) + W_s(t_2)$ represent $\succeq_t$ on $\mathcal{T}^2_\downarrow$. By the uniqueness properties of U_r and U_s, there exist real q_r, q_s and positive $V(q)$ such that for all $t \in \mathcal{T}$:

$$U_r(q, t) = V(q)W_r(t) + q_r \tag{A1}$$

$$U_s(q, t) = V(q)W_s(t) + q_s. \tag{A2}$$

Set $W_r(0) = W_s(0) = 0$, which is allowed by the uniqueness properties of U_r and U_s and the zero-condition. Now, $0 = W_r(0) = W_s(0) = U_r(q', 0) = U_s(q', 0) = U_r(q, 0) = U_s(q, 0)$ where the latter two equalities follow by the zero-condition. Therefore, $V(q)\cdot 0 + q_r = 0$ from which it follows that for all $q \in \mathcal{Q}$, $q_r = 0$. Similarly it can be shown that for all $q \in \mathcal{Q}$, $q_s = 0$. By outcome monotonicity, $U_r(q, t_1) = V(q)W_r(t_1)$ and $U_s(q, t_1) = V(q)W_s(t_1)$ represent the same preference relation on $\mathcal{H} = \mathcal{Q} \times \mathcal{T}$. Hence, by the uniqueness properties of multiplicative representations (Krantz *et al.*, 1971) there exist $\lambda,\gamma > 0$ such that for all $(q,t) \in \mathcal{H}, U_r(q, t) = V(q)W_r(t) = \lambda V(q)^\gamma W_s(t)^\gamma = \lambda U_s(q, t)^\gamma$.

Because not all health states are equivalent, $V(q)$ is not constant. Let $q_1, q_2 \in \mathcal{Q}$ be such that $V(q_1) \neq V(q_2)$. Then for arbitrary $t \in \mathcal{T}, t \neq 0, V(q_1)W_r(t) = \lambda\cdot V(q_1)^\gamma W_s(t)^\gamma$ and $V(q_2)W_r(t) = \lambda V(q_2)^\gamma W_s(t)^\gamma$. Substitution gives $\frac{V(q_1)}{V(q_2)} = \left[\frac{V(q_1)}{v(Q_2)}\right]^\gamma$ Hence, $\gamma = 1$ and U_r and U_s are linear with respect to each other. Define $W(t) = W_s(t), U(q,t) = V(q)W(t), \pi_r = \frac{\lambda}{\lambda+1}$ and $\pi_s = \frac{1}{\lambda+1}$. Then the nonlinear QALY model holds and $\pi_r U$ and $\pi_s U$ are additive utility functions for $\succeq$ on $\mathcal{H}^2_\downarrow$.

Appendix E: Question Wording in the First Experiment.

Suppose that you have been diagnosed to have symptoms of one of two diseases: *A* or *B*. From medical experience it is known that half of the people with these symptoms have disease *A* and half have disease *B*. There exist two treatments for these diseases but the effects of the treatments depend on which disease you have. To be effective, the treatments have to start immediately. Unfortunately, it is only known which disease you have after treatment has started. That is, you have to choose which treatment to undergo when you are still uncertain which disease you have.

In the following questions you are faced with different outcomes of the treatments. In each question you are asked to state the number of life years for which you consider the two treatments equivalent. Suppose in every question that you spend the years in good health. The way of presentation is as follows:

Treatment	Disease *A*	Disease *B*
1	50	5
2	25	x_i

Suppose you choose to undergo treatment 1, then you live for 50 more years if you turn out to have disease *A*. If, on the other hand, you turn out to have disease *B* you live for 5 more years. If you choose to undergo treatment 2 you live for 25 more years if you turn out to have disease *A*. If, on the other hand, you turn out to have disease *B* you live for x_i more years. In the following questions you will be asked to indicate the number of years x_i for which you consider the two treatments equivalent. In the above example, it is plausible that x_i lies between 5 years and 50 years. If $x_i = 5$ years then treatment 1 is clearly better than treatment 2. If $x_i = 50$ years then treatment 2 is clearly better than treatment 1.

Now consider the following question:

Question 1.

If you choose treatment 1 and you turn out to have disease *A* you live for 55 more years in good health, but if you turn out to have disease *B* you die immediately. If you choose treatment 2 and you turn out to have disease *A* you live for 45 more years in good health, but if you turn out to have disease *B* you live for x_1 more years in good health. Choose the value of x_1 for which you consider the two treatments equivalent and put this value on your answer sheet.

Treatment	Disease *A*	Disease *B*
1	55	0
2	45	x_1

Appendix F: The Description of the Health States in the Second Experiment

Back Pain
Unable to perform some tasks at home and/or at work Able to perform all self care activities (eating, bathing, dressing) albeit with some difficulties Unable to participate in many types of leisure activities Often moderate to severe pain and/or other complaints
Migraine
Unable to perform usual tasks at home and/or at work Able to perform all self care activities (eating, bathing, dressing) albeit with some difficulties Unable to participate in any type of leisure activity Severe headache